Bipolarisation measurement: A two-income approach with an application to nine Sub-Saharan African countries

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September 28, 2023

Income and wealth inequality: drivers and consequences

Bipolarisation?

Literature on measurement initiated in the 1990s by Michael Wolfson (Foster and Wolfson, 2010, Wolfson, 1994, Duclos et al., 2004). Here only bipolarisation is considered. Bipolarisation: Tendency of observed values to group into two homogeneous distinct groups. Bipolarisation means partitioning income distributions into two parts (bottom and top). Pigou-Dalton transfers have different effects depending on the ranks of the donor and the recipient:

- from top part to bottom part $\Rightarrow$ bipolarisation decreases,
- within bottom part $\Rightarrow$ bipolarisation increases,
- within top part $\Rightarrow$ bipolarisation increases.

Bipolarisation vs inequality

Contrast with inequality indices that suppose progressive transfer always reduce inequality (Dalton, 1920, Dasgupta et al., 1973).

Moreover, reference situations partly shared:

- perfect equality $=$ minimum bipolarisation $=$ minimum inequality,
- maximum bipolarisation $\neq$ maximum inequality.

E.g. 4-person population with total income equals 10 and assuming partition into two halves:

- $(0, 0, 5, 5) \Rightarrow$ maximum bipolarisation,
- $(0, 0, 0, 10) \Rightarrow$ maximum inequality.

Why should we care about bipolarisation?

General concern with the existence of a middle class. Both intrinsic and instrumental reasons: ensuring social cohesion, broaden the tax base, foster entrepreneurship, show resilience to negative shocks...

Official goal in China:

> *[...] at a prominent and widely-reported meeting in August 2021 of the Chinese Community Party's Central Committee for Financial and Economic Affairs, President Xi Jinping argued that China needed a more "olive-shaped distribution structure of large middle and small ends." (Ravallion and Chen, 2022, p.1)*

Current state of art I

Numerous contributions:

- Measurement: Foster and Wolfson (2010), Wang and Tsui (2000), Chakravarty and Majumder (2001), Rodríguez and Salas (2003), Silber et al. (2007), Lasso de la Vega et al. (2010), Makdissi and Mussard (2011), Yalonetzky (2017), Ciommi et al. (2022),
- Axiomatic and robustness: Chakravarty and Majumder (2001), Duclos and Échevin (2005), Bossert and Schworn (2008), Chakravarty and D'Ambrosio (2010)
- Reviews: Chakravarty (2009), Permanyer (2018),
- Experimentation: Amiel et al. (2010),
- Empirical applications: Ravallion (2010), Roope et al. (2018), Ravallion and Chen (2022).

Current state of art II

But proposed indices:

- lack flexibility regarding the bottom/top partition,
- lack theoretical grounds (social welfare functions),
- have no intuitive interpretation.

Contributions

Here, two new families of bipolarisation indices:

- with intuitive pie-sharing representation,
- not limited to 50/50 partition,
- linked with usual inequality indices,
- perfectly decomposable into spread and clustering components.

Illustrations using consumption series from 2018–2019 EHCVM (Benin, Burkina Faso, Chad, Cote d'Ivoire, Guinea Bissau, Mali, Niger, Senegal, and Togo).

Notations I

Let:

- $n \in \mathbb{N}^* \setminus \{1\}$: population size,
- $N := \{1, \dots n\}$: set of individuals,
- $x_i \in \mathbb{R}_+$: individual $i \in N$ income,
- $X := (x_1, \dots, x_n)$: income vector of N,
- $\mathcal{D}^n := \{X \in \mathbb{R}_+^n | \exists x_i > 0\}$ et $\mathcal{D} := \cup_{n=2}^{+\infty} \mathcal{D}^n$,
- μ_X: mean value of X,
- $r : N \to N$: rank function (in increasing order),
- $p \in]0; 1[$: a population percentage,

Notations II

- $\underline{X}$: income vector for the bottom part X,
- $\bar{X}$: income vector for the top part of X,
- $\Psi : \mathcal{D} \to \mathbb{R}$: a bipolarisation index.

Quasi-linear means I

Quasi-linear means defined by:

$$\tau_X^W := \rho^{-1}\left(\sum_{i=1}^{n} w_i \rho(x_i)\right),$$

where $W := (w_1, \ldots w_n)$ is a vector of strictly positive weights and $\rho : \mathbb{R} \to \mathbb{R}$ is bijective. Here focus on:

$$\tau_X^{\gamma,W} := \begin{cases} \frac{1}{\gamma}\log\left(\sum_{i=1}^{n} w_i e^{\gamma x_i}\right) & \text{if } \gamma \neq 0 \\ \sum_{i=1}^{n} w_i x_i & \text{if } \gamma = 0 \end{cases},$$

$$\tau_X^{\alpha,W} := \begin{cases} \left(\sum_{i=1}^{n} w_i x_i^{\alpha}\right)^{\frac{1}{\alpha}} & \text{if } \alpha \neq 0 \\ \prod_{i=1}^{n} x_i^{w_i} & \text{if } \alpha = 0 \end{cases},$$

with $\sum_{i=1}^{n} w_i = 1$ and $w_i \geqslant 0 \forall i \in N$.

Quasi-linear means II

Quasi-linear means include:

- arithmetic means for $\alpha = 1$ or $\gamma = 0$, and $w_i = \frac{1}{n}\ \forall i \in N$,
- quadratic means for $\alpha = 2$ and $w_i = \frac{1}{n}\ \forall i \in N$,
- geometric means for $\alpha = 0$ and $w_i = \frac{1}{n}\ \forall i \in N$,
- harmonic means for $\alpha = -1$ and $w_i = \frac{1}{n}\ \forall i \in N$,
- S-Gini for $\alpha = 1$ and $w_i = \frac{1}{n^\delta}\left(\left(n+1-r(i,X)\right)^\delta - \left(n-r(i,X)\right)^\delta\right)$ (Gini for $\delta = 2$).

Estimating bipolarisation with quasi-linear means: absolute indices

We suggest:

$$\hat{\Psi}^A(X;p) := \tau_{\bar{X}}^{\gamma',W'} - \tau_{\underline{X}}^{\gamma,W},$$

For $\hat{\Psi}^A$ to satisfy desired properties for a bipolarisation index (axioms), we need:

- for $\underline{\tau}$, we have $\gamma \leqslant 0$ and $w_i = \pi\big(r(i,\underline{X})\big)$ where π is a non-decreasing function of r,
- for $\bar{\tau}$, we have $\gamma \geqslant 0$ and $w_i = \pi\big(r(i,\bar{X})\big)$ where π is a non-increasing function of r.

First family of indices suggested by Wang and Tsui (2000) obtained for $\gamma = 0$.

Estimating bipolarisation with quasi-linear means: relative indices

We suggest:

$$\hat{\Psi}^R(X;p) := \frac{(1-p)\left(\tau_{\bar{X}}^{\alpha',W'} - \tau_{\underline{X}}^{\alpha,W}\right)}{\eta},$$

with $\eta := p\tau_{\underline{X}}^{\alpha,W} + (1-p)\tau_{\bar{X}}^{\alpha',W'}$. For $\hat{\Psi}^R$ to satisfy desired properties for a bipolarisation index (axioms), we need:

- for $\underline{\tau}$, we have $\alpha \geqslant 1$ and $w_i = \pi\left(r(i,\underline{X})\right)$ where π is a non-decreasing function of r,
- for $\bar{\tau}$, we have $\alpha \leqslant 1$ and $w_i = \pi\left(r(i,\bar{X})\right)$ where π is a non-increasing function of r.

Second family of indices suggested by Wang and Tsui (2000) obtained for $\alpha = 1$.

Two poles!

No successful definition of bipolarisation indices with the help of the EDE approach.
⇒ Instead of looking for a measure based on a single EDE income, why not using two representative incomes?
After all, it's bi-polar-isation!

The idea

For given X and p, let assume there exists $(\underline{x}, \bar{x}) \in \mathbb{R}_+^2$ (with $p\underline{x} + (1-p)\bar{x} = \mu$) that is representative of $(\underline{X}, \bar{X})$, *i.e.* pn persons all having $\underline{x}$ and $(1-p)n$ persons receiving $\bar{x}$ would result in the same level of bipolarisation as $(\underline{X}, \bar{X})$.
Reminiscent of the "dichotomously allocated equivalent distribution" suggested by Subramanian (2002, 2013),
Then:

- $\Psi_p^A := \bar{x} - \underline{x}$ is an absolute measure of bipolarisation,
- $\Psi_p^R := \frac{\bar{s}_p - (1-p)}{p}$, with $\bar{s}_p$ the income share of the top part, is a relative measure of bipolarisation.

A condition and some remarks

Ψ_p^A and Ψ_p^R can be used as bipolarisation indices provided mean-preserving spreads within $\underline{X}$ or $\bar{X}$ imply $\mu_{\underline{X}} \leqslant \underline{x} \leqslant \bar{x} \leqslant \mu_{\bar{X}}$.

With the considered hypotheses:

- $\Psi_p^A = \Psi_p^R = 0$ if $\underline{x} = \bar{x}$,
- $\Psi_p^R = 1$ if $\underline{x} = 0$,
- $\hat{\Psi}^R = \frac{\bar{s}_p - (1-p)}{p} = \frac{(1-p)(\bar{x}-\underline{x})}{p\underline{x}+(1-p)\bar{x}}$,

Spread-clustering decomposition

Indices can easily be decomposed into:

$$\Psi_p^A = \underbrace{(\mu_{\bar{X}} - \mu_{\underline{X}})}_{\geqslant 0} + \underbrace{((\bar{x} - \mu_{\bar{X}}) + (\mu_{\underline{X}} - \underline{x}))}_{\leqslant 0},$$

$$\Psi_p^R = \underbrace{\frac{(1-p)(\mu_{\bar{X}} - \mu_{\underline{X}})}{\mu}}_{\geqslant 0} + \underbrace{\left(\frac{(1-p)(\bar{x} - \mu_{\bar{X}})}{\mu} + \frac{(1-p)(\mu_{\underline{X}} - \underline{x})}{\mu}\right)}_{\leqslant 0}.$$

Relationship with proposed indices

We show that:

- Assuming $\left(\underline{\tau}_{\underline{X}}^{\gamma,W}, \bar{\tau}_{\bar{X}}^{\gamma',W'}\right)$ shows the same level of bipolarisation as $(\underline{x}, \bar{x})$, $\hat{\Psi}^{A}(X;p)$ is a valid estimator of Ψ_p^A with $\underline{x} = \underline{\tau}_{\underline{X}}^{\gamma,W} - (\eta - \mu)$ and $\bar{x} = \bar{\tau}_{\bar{X}}^{\gamma',W'} - (\eta - \mu)$.
- Assuming $\left(\underline{\tau}_{\underline{X}}^{\alpha,W}, \bar{\tau}_{\bar{X}}^{\alpha',W'}\right)$ shows the same level of bipolarisation as $(\underline{x}, \bar{x})$, $\hat{\Psi}^{R}(X;p)$ is a valid estimator of Ψ_p^R with $\underline{x} = \frac{\mu_X}{\eta}\underline{\tau}_{\underline{X}}^{\alpha,W}$ and $\bar{x} = \frac{\mu_X}{\eta}\bar{\tau}_{\bar{X}}^{\alpha',W'}$.

Quasi-linear means and the AKS approach

Inequality indices Θ from the Atkinson-Kolm-Sen (AKS) approach linked to quasi-linear means. We show how $\hat{\Psi}^A(X;p)$ and $\hat{\Psi}^R(X;p)$ can be expressed as functions of $\mu_{\underline{X}}$, $\mu_{\bar{X}}$, known inequality indices (Gini, Generalized Gini, Atkinson, Kolm-Pollack...), and new inequality indices (details):

$$\hat{\Psi}^A(X;p) = (\mu_{\bar{X}} - \Theta'_{\bar{X}}) - (\mu_{\underline{X}} - \Theta_{\underline{X}}),$$

$$\hat{\Psi}^R(X;p) = \frac{(1-p)\big((1-\Theta'_{\bar{X}})\mu_{\bar{X}} - (1+\Theta_{\underline{X}})\mu_{\underline{X}}\big)}{\eta},$$

with $\eta = p(1+\Theta_{\underline{X}})\mu_{\underline{X}} + (1-p)(1-\Theta'_{\bar{X}})\mu_{\bar{X}}$.

The Gini weighing scheme

Using the weighing scheme of the Gini index G for τ, we obtain:

$$\hat{\Psi}^{AG}(X;p) = (1 - G_{\bar{X}})\mu_{\bar{X}} - (1 + G_{\underline{X}})\mu_{\underline{X}},$$

$$\hat{\Psi}^{RG}(X;p) = \frac{(1-p)\big((1 - G_{\bar{X}})\mu_{\bar{X}} - (1 + G_{\underline{X}})\mu_{\underline{X}}\big)}{\eta},$$

with $\eta = p(1 + G_{\underline{X}})\mu_{\underline{X}} + (1-p)(1 - G_{\bar{X}})\mu_{\bar{X}}$. (FW)

Data

Consumption series from 2018–2019 *Enquête Harmonisées sur les Conditions de Vie des Ménages* (EHCVM) 2018–2019 for Benin, Burkina Faso, Chad, Côte d'Ivoire, Guinea Bissau, Mali, Niger, Senegal, and Togo.

Many possible choices for p like:

- the headcount index,
- 20%, 40% (SDG10, World Bank's twin goals),
- 50% (usual cutoff for bipolarisation analysis),
- 90%, 95%, 99%...

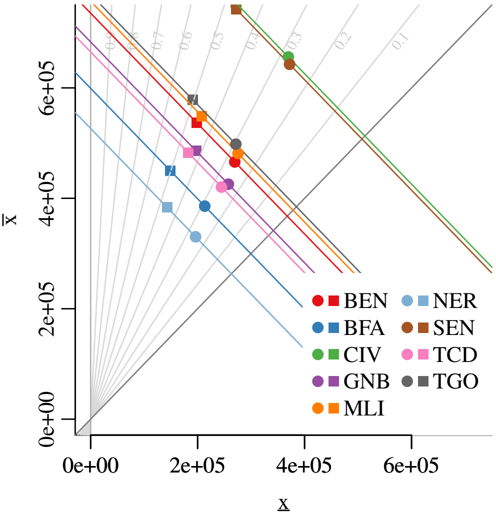
Table 1: Mean consumption and relative inequality levels in selected Sub-Saharan African countries, 2018–2019.

	Mean		Gini		Atk. ($\alpha = 0.5$)		Atk. ($\alpha = 0$)	
	μ	rank	Θ	rank	Θ	rank	Θ	rank
Guinea Bissau	3.42e+05	4	0.316	1	0.0817	1	0.149	1
Mali	3.78e+05	6	0.332	2	0.0879	2	0.163	2
Chad	3.33e+05	3	0.336	3	0.0918	3	0.169	3
Benin	3.68e+05	5	0.347	4	0.0984	4	0.179	4
Niger	2.63e+05	1	0.35	5	0.103	7	0.182	5
Senegal	5.07e+05	8	0.351	6	0.103	6	0.183	7
Côte d'Ivoire	5.13e+05	9	0.351	7	0.0996	5	0.183	6
Togo	3.85e+05	7	0.381	8	0.119	8	0.214	8
Burkina Faso	3e+05	2	0.386	9	0.124	9	0.216	9

Note: Atk denotes the Atkinson inequality index.

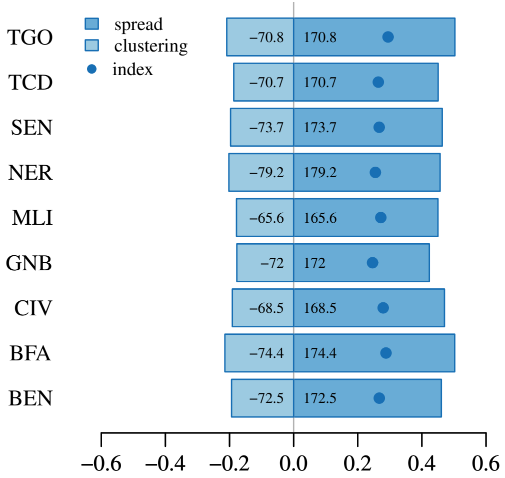
Table 2: Relative bipolarisation levels in selected Sub-Saharan African countries, 2018–2019.

	$\underline{\alpha}=\bar{\alpha}=1$; $\underline{\delta}=\bar{\delta}=2$		$\underline{\alpha}=\bar{\alpha}=1$; $\underline{\delta}=\bar{\delta}=3$		$\underline{\alpha}=2;\bar{\alpha}=0.5$; $\underline{\delta}=\bar{\delta}=1$		$\underline{\alpha}=3;\bar{\alpha}=0$; $\underline{\delta}=\bar{\delta}=1$	
	$\hat{\Psi}^R$	rank	$\hat{\Psi}^R$	rank	$\hat{\Psi}^R$	rank	$\hat{\Psi}^R$	rank
				$p=0.5$				
Guinea Bissau	0.246	1	0.176	1	0.39	1	0.361	1
Mali	0.272	6	0.196	6	0.417	4	0.388	4
Chad	0.264	3	0.188	3	0.414	2	0.382	3
Benin	0.267	5	0.189	4	0.421	5	0.388	5
Niger	0.255	2	0.179	2	0.414	3	0.379	2
Senegal	0.267	4	0.19	5	0.422	6	0.388	6
Côte d'Ivoire	0.279	7	0.2	7	0.432	7	0.399	7
Togo	0.294	9	0.209	9	0.457	9	0.419	9
Burkina Faso	0.288	8	0.204	8	0.455	8	0.415	8
				$p=0.9$				
Guinea Bissau	0.0942	1	0.0679	1	0.15	1	0.132	1
Mali	0.1	2	0.0725	2	0.154	2	0.135	2
Chad	0.102	3	0.0733	3	0.16	3	0.139	3
Benin	0.107	4	0.0769	4	0.168	5	0.146	4
Niger	0.112	7	0.0789	6	0.182	7	0.157	7
Senegal	0.11	6	0.0792	7	0.176	6	0.152	6
Côte d'Ivoire	0.108	5	0.0785	5	0.168	4	0.146	5
Togo	0.119	8	0.0852	8	0.187	8	0.159	8
Burkina Faso	0.129	9	0.0919	9	0.203	9	0.174	9



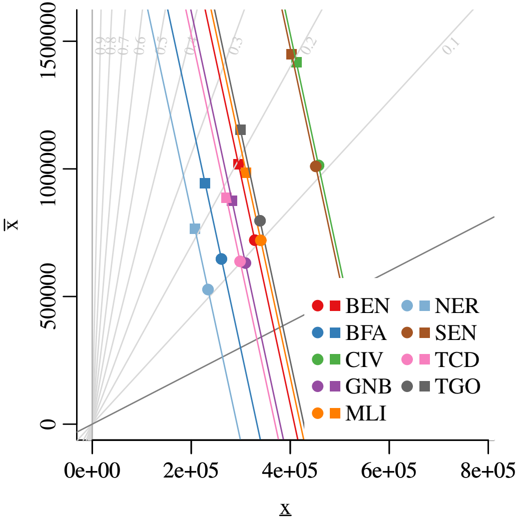
Note: Bullets depicts $(\underline{x}, \bar{x})$ while squares are for $(\mu_{\underline{X}}, \mu_{\bar{X}})$.

Figure 1: Two income representation for nine Sub-Saharan countries, 2018–2019: $p = 0.5$.



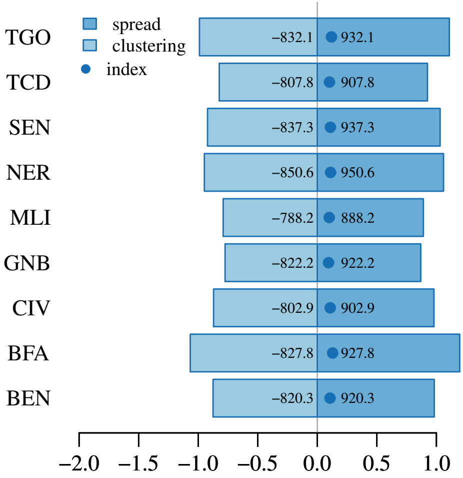
Note: Figures within bar indicates the size, in percentage, of the clustering and spread components relative to the value of the bipolarisation index.

Figure 2: Spread and clustering components of $\hat{\Psi}^{RG}$ for selected Sub-Saharan African countries, 2018–2019: $p = 0.5$.



Note: Bullets depicts $(\underline{x}, \bar{x})$ while squares are for $(\mu_{\underline{X}}, \mu_{\bar{X}})$.

Figure 3: Two income representation for nine Sub-Saharan countries, 2018–2019: $p = 0.9$.



Note: Figures within bar indicates the size, in percentage, of the clustering and spread components relative to the value of the bipolarisation index.

Figure 4: Spread and clustering components of $\hat{\Psi}^{RG}$ for selected Sub-Saharan African countries, 2018–2019: $p = 0.9$.

Final remarks - To Do

- Polishing proofs (and the rest of the paper),
- Standard error formula,
- More empirical evidence,
- Ground indices on SWF.

The end

Thank you for your attention!

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Additional notations

- $b(X;p) := \left\{ i \in N \,\middle|\, \frac{r(i,X)}{n} \leqslant p \right\}$: the worst off,
- $t(X;p) := \left\{ i \in N \,\middle|\, \frac{r(i,X)}{n} > p \right\}$: the better off with $b(X;p) \cup t(X;p) = N$ and $b(X;p) \cap t(X;p) = \emptyset$,
- $\mathcal{O}^n$: vector of zero incomes,
- $\mathcal{E}^n$: perfectly egalitarian distribution, and $\mathcal{E} := \cup_{n=1}^{+\infty} \mathcal{E}^n$,
- $\mathcal{B}_p^n := \{(x,y) \subset \mathcal{D}^n | x = \mathcal{O}^{pn} \;\&\; y \in \mathcal{E}^{(1-p)n}\}$: a maximum bipolarisation distribution,
- $\mathcal{P}^n$: set of $n \times n$ permutation matrices,
- $c(X,Y) :=$ $\{i \in N \,|\exists j \in N$ s. t. $r(j,X) = r(i,Y) \wedge y_j = x_i\}$: set of individuals in X whose income also exists in Y at the same rank,
- $\tilde{c}(X,Y)$: complement of $c(X,Y)$.

Axiomatic framework: Spread

Spread increasing transfer (SPT): $\forall n \in \mathbb{N}^* \setminus \{1\}$, $X \in \mathcal{D}^n$, $p \in]0,1[$, $i \in b(X;p)$, $j \in t(X;p)$, and $\delta \in]0, x_i]$, we have $\Psi(Y,p) \geqslant \Psi(X,p)$ if $y_i = x_i - \delta$, $y_j = x_j + \delta$, and $y_k = x_k \; \forall k \in N \setminus \{i,j\}$.

Spread increasing changes (SPC): $\forall n \in \mathbb{N}^* \setminus \{1\}$, $X \in \mathcal{D}^n$, $p \in]0,1[$, we have $\Psi(Y,p) \geqslant \Psi(X,p)$ if $y_i = x_i + \delta$, $y_k = x_k \; \forall k \in N \setminus \{i\}$ and either:

i) $i \in b(X;p)$ and $\delta \in [-x_i, 0]$, or
ii) $i \in t(X;p)$ and $\delta \in \mathbb{R}_+$.

Axiomatic framework: Clustering

Clustering increasing transfer (CLU): $\forall n \in \mathbb{N}^* \setminus \{1,2\}$, $X \in \mathcal{D}^n$, $p \in]0,1[$, $x_i < x_j$ and $\delta \in \left]0, \frac{x_j - x_i}{2}\right]$, we have $\Psi(Y,p) \geqslant \Psi(X,p)$ if either:

i) $pn \geqslant 2$, $\{i,j\} \subseteq b(X;p)$, $y_i = x_i + \delta$, $y_j = x_j - \delta$, and $y_k = x_k \; \forall k \in N \setminus \{i,j\}$, or

ii) $(1-p)n \geqslant 2$, $\{i,j\} \subseteq t(X;p)$, $y_i = x_i + \delta$, $y_j = x_j - \delta$, and $y_k = x_k \; \forall k \in N \setminus \{i,j\}$.

back

Axiomatic framework: Independence

Weak Independence (IND_W): $\forall n \in \mathbb{N}^* \setminus \{1\}$,
$\{X, X', Y, Y'\} \subset \mathcal{D}^n$ such that $r(i,X) = r(i,Y)$ $\forall i \in c(X,Y)$, we observe $\Psi(X,p) \geqslant \Psi(Y,p) \Leftrightarrow \Psi(X',p) \geqslant \Psi(Y',p)$ if $r(i,X') = r(i,Y')$ $\forall i \in c(X,Y)$, $r(i,X) = r(i,X')$ $\forall i \in \tilde{c}(X,Y)$ and $r(i,Y) = r(i,Y')$ $\forall i \in \tilde{c}(X,Y)$.

Strong Independence (IND_S): $\forall n \in \mathbb{N}^* \setminus \{1\}$, $s \in N \setminus \{n\}$,
$\{A, B\} \subset \mathcal{D}^{n-s}$, $\{C, D\} \subset \mathcal{D}^s$, $X = (A, C)$, $X' = (A, D)$, $Y = (B, C)$, and $Y' = (B, D)$, we observe $\Psi(X,p) \geqslant \Psi(Y,p) \Leftrightarrow \Psi(X',p) \geqslant \Psi(Y',p)$ if:

i) $b(X,p) \cap c(X,X') = b(X',p) \cap c(X,X')$,
ii) $t(X,p) \cap c(X,X') = t(X',p) \cap c(X,X')$,
iii) $b(Y,p) \cap c(Y,Y') = b(Y',p) \cap c(Y,Y')$, and

Axiomatic framework: Independence

Let's consider $A = (4, 8, 10, 20)$, $B = (7, 8, 10, 18)$, $p = \frac{1}{2}$, and assume $\Psi(A) < \Psi(B)$.

- With $A^* = (4, 11, 16, 20)$ and $B^* = (7, 11, 16, 18)$, both IND$_S$ and IND$_W$ $\Rightarrow \Psi(A^*) < \Psi(B^*)$,
- With $A' = (4, 6, 12, 20)$ and $B' = (6, 7, 12, 18)$, IND$_S$ $\Rightarrow \Psi(A') < \Psi(B')$, but IND$_W$ $\Rightarrow \Psi(A') \gtreqless \Psi(B')$,
- With $A'' = (4, 5, 6, 20)$ and $B'' = (5, 6, 7, 18)$, both IND$_S$ and IND$_W$ $\Rightarrow \Psi(A'') \gtreqless \Psi(B'')$.

back

Axiomatic framework: Other fundamental axioms

Anonymity (ANO): $\forall n \in \mathbb{N}^*$, $X \in \mathcal{D}^n$, $P \in \mathcal{P}^n$, we have
$\Psi(PX, p) = \Psi(X, p)$.

Population (POP): $\forall \{n, \lambda\} \subset \mathbb{N}^*$, $X \in \mathcal{D}^n$, we have
$\Psi\big(d_\lambda(X), p\big) = \Psi(X, p)$.

Unit consistency (UNC): $\forall Y, X \in \mathcal{D}$, $p \in]0, 1[$,
$\Psi(Y, p) \geqslant \Psi(X, p) \Leftrightarrow \Psi(\kappa Y, p) \geqslant \Psi(\kappa X, p)$
$\forall \kappa \in \mathbb{R}_{++}$.

back

Axiomatic framework: Additional axioms I

Continuity (CON): $\forall n \in \mathbb{N}^*$, Ψ has continuous first-order partial derivatives $\frac{\partial \Psi}{\partial x_i}$ over $\mathcal{D}^n$.

Scale invariance (SCI): $\forall X \in \mathcal{D}$, $p \in]0,1[$, and $\kappa \in \mathbb{R}_{++}$, we have $\Psi(\kappa X, p) = \Psi(X, p)$.

Translation invariance (TRI): $\forall n \in \mathbb{N}^*$, $X \in \mathcal{D}^n$, $E \in \mathcal{E}^n$, and $p \in]0,1[$, we have $\Psi(X + E, p) = \Psi(X, p)$.

Equality normalisation (ENO): $\forall p \in]0,1[$, $\Psi(X, p) = 0$ if and only if $X \in \mathcal{E}$.

Maximum bipolarity normalisation (MNO): $\forall p \in]0,1[$, $\Psi(X, p) = 1$ if $X \in \mathcal{B}_p$ and $n \in \mathbb{N}^* \setminus \{1\}$.

back

Axiomatic framework: Additional axioms II

Linearity (LIN): $\forall n \in \mathbb{N}^* \setminus \{1\}$, $p \in]0,1[$, $\alpha \in [0,1]$, $X \in \mathcal{E}^n$, $Y \in \mathcal{B}_p^n$, such that $\mu_X = \mu_Y$, we have $\Psi(Z,p) = \alpha\Psi(X,p) + (1-\alpha)\Psi(Y,p)$ if $Z = \alpha X + (1-\alpha)Y$.

back

Extended AKS framework for inequality indices

Generalization of Kolm, Atkinson and Sen approach for inequality measurement based on quasi-linear means and SWF. Noting x^E an EDE income, we can define normative indices as:

$$\Theta^A := \begin{cases} \mu - x^E & \text{with } x^E \text{ being S-concave} \\ x^E - \mu & \text{with } x^E \text{ being S-convex} \end{cases},$$

$$\Theta^R := \begin{cases} 1 - \frac{x^E}{\mu} & \text{with } x^E \text{ being S-concave} \\ \frac{x^E}{\mu} - 1 & \text{with } x^E \text{ being S-convex} \end{cases}.$$

Extended AKS framework: new indices

Complement D of Atkinson's indices, that is:

$$D(X;\alpha) := \left(\frac{1}{n}\sum_{i=1}^{n}\left(\frac{x_i}{\mu_X}\right)^{\alpha}\right)^{\frac{1}{\alpha}} - 1, \quad \text{with } \alpha > 1.$$

back

The Foster-Wolfson bonus

We have:

$$\hat{\Psi}^{AG}(X;p) = 2y(0.5)FW = 2\mu_X FW_\mu,$$
$$\hat{\Psi}^{RG}(X;p) = \frac{FW_\mu}{1 + 2\underline{G}^W - 2\bar{G}^W},$$

with $FW := (G^B - G^W)\frac{\mu}{y(0.5)}$ being the Fost-Wolfson index, FW_μ its mean-normalized version, $\underline{G}^W = \frac{\mu_{\underline{X}}}{4\mu}G_{\underline{X}}$ and $\bar{G}^W = \frac{\mu_{\bar{X}}}{4\mu}G_{\bar{X}}$

back