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Revisiting poverty measures using quantile functions

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Abstract

In this article we redefine various poverty measures in literature in terms of quantile functions instead of distribution functions in the prevailing approach. This enables provision for alternative methodology for poverty measurement and analysis along with some new results that are difficult to obtain in the existing framework. Several flexible quantile function models that can enrich the existing ones are proposed and their utility is demonstrated for real data.

Keywords: Poverty measures, Sen index, quantile function, income models.

1. Introduction

Poverty, interpreted as a state or condition in which an individual or community lacks resources and other essentials for a minimum standard of living, is a widely discussed subject in socio-economic literature. As a matter of concern for alleviation of poverty through various welfare programmes, there is need to identify the poor and to understand the intensity of poverty. Towards this objective, several measures of poverty such as head count ratio, poverty (income) gap ratio, Gini index for the poor, etc., and their combinations in the form of various indices like the Sen index and the FGT index satisfying certain axioms of behaviour have been proposed in the last few decades. An essential feature of all these indices of poverty is that they are based on the income distribution of the population, despite the recent view that poverty is related to social welfare which includes other facts than income inequality. See Yang (2017) for a discussion on the relationship between poverty and income inequality.

The definition of poverty measures based on income distribution, uses the distribution function of incomes in specifying them. Although probability distributions can also be specified by quantile functions, as an alternative equivalent of distribution functions, the prospect of looking at poverty measures using quantile functions is a rarely visited area in econometrics. The present work is an attempt

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to redefine the various poverty concepts in the framework of quantile functions and to highlight the additional advantages arising therefrom. In the first place it gives a better insight into the properties of poverty indices by obtaining new results that are difficult to achieve by employing the distribution functions. Many quantile functions do not have tractable distribution functions that limits their practical utility in the existing distribution function based framework. The fact that they are flexible to contain some of the existing income models as special cases and approximates many others satisfactorily, points out the need to accommodate them also as candidate models in practical problems. In addition to these, generally quantile functions possess properties specific to modelling problems, that are not satisfied by distribution functions. We refer to Gilchrist (2000) for details.

Besides providing the quantile-based definitions of the poverty (income) gap ratio, mean function, Gini index of the poor, Sen and Sen-Shorrocks indices, we show that the first four determines the income law and conditions under which they can be chosen. The relationship between Lorenz curve and the poverty measures are also found. Some highly flexible quantile function that represent the generalized lambda, Wakeby, Kappa and Govindarajulu distributions are proposed as prospective income models and their poverty measures are presented. the utility of our results are demonstrated for income data pertaining to the California state.

The paper is organized into five sections. After the present introductory section, in Sections 2 and 3 we discuss the additive separable measures and rank-based measures of poverty. In Section 4 we propose some flexible quantile functions as income models. Finally, the applications of our results in analysing data on incomes of the California state are presented in Section 5.

2. Additively separable measures

In this section we present the definitions and properties of a class of poverty measures that are additively separate, using quantile functions. They serve as a set of alternative methodologies to analyse the poverty in a population. In our knowledge, the properties mentioned in the Theorems have not appeared previously in literature.

Let X be a continuous non-negative random variable representing the income in a population with distribution function $F(x) = P(X \leq x)$, probability density function $f(x)$ and quantile function

$$Q(u) = \inf[x \mid F(x) \geq u], \quad 0 \leq u \leq 1.$$

When $F(x)$ is continuous and strictly increasing, $F(Q(u)) = u$, so that on differentiation

$$f(Q(u)) = [q(u)]^{-1}$$

where $q(u) = \frac{dQ(u)}{du}$ is the quantile density function and $F(x) = u$ if and only if $x = Q(u)$.

We assume that $x = t$ is the poverty line so that all individuals with income below t are considered poor and that there exist a function representing the poverty of the individual. A general approach (Atkinson (1987), Hagenaars (1987)) to the definition of a poverty measure is to propose an additively separable aggregate poverty metric

$$A_F(t) = \int_0^t a(x,t)dF(x) \tag{2.1}$$

with $a(x,t)$ interpreted as a deprivation suffered with income x satisfying $a(x,t) = 0$ for $x \geq t$. Many poverty measures discussed in literature like those of Watts (1969), Thon (1979), Clark et al. (1981), etc., belong to this class. When expressed in terms of quantile functions (2.1) has the form

$$A_Q(u) = \int_0^u a(Q(p), Q(u))dp. \tag{2.2}$$

An important class of measures suggested by Foster et al. (1984) reads

$$A_F(t,\alpha) = \int_0^t \left(1 - \frac{x}{t}\right)^\alpha dF(x) \tag{2.3}$$

or equivalently

$$A_Q(Q(u),\alpha) = A_Q(u,\alpha) = \int_0^u \left(1 - \frac{Q(p)}{Q(u)}\right)^\alpha dp. \tag{2.4}$$

When $\alpha = 0$, (2.3) reduces to $F(t)$ which is the head count ratio giving the proportion of poor in the population. Another important special case, discussed in many contexts is obtained when $\alpha = 1$,

$$A_Q(u,1) = A_1(u) = u - \frac{1}{Q(u)} \int_0^u Q(p)dp \tag{2.5}$$

called the poverty gap ratio (PGR). Note that u is the head-count ratio and PGR is interpreted as the shortfall of the income of the poor from the poverty line and indicates the poverty line and the average income of the poor. Some new properties of PGR and their implications to poverty analysis are presented below.

Theorem 2.1. *The income distribution is uniquely determined by $A(u)$ through the formula*

$$Q(u) = \frac{\exp\left[-\int_u^1 (p - A_1(p))^{-1} dp\right]}{u - A_1(u)}. \tag{2.6}$$

The proof of the theorems are given in the appendix. Some implications of Theorem 2.1 are the following.

Theorem 2.1 says that if the functional form of $A_1(u)$ is known or obtained by considering the empirical form of $A_1(u)$, then the income distribution can be found directly from (2.6) (see Section 5). The expression of $A_1(u)$ for several quantile functions are exhibited in Table 1 for easy reference to the corresponding distribution.

A large number of distributions prescribed in literature as models of income, were not based on a clear rationale, but simply justified by the fact they provided satisfactory goodness of fit. In modelling, it is a widely accepted fact that some distinct features of data must form the basis of selecting the model. Characterization theorems have a great role in finalising the appropriate model. Taking these into consideration by fixing appropriate PGR for the data we have a criterion to choose the model as well as to give sufficient representation for empirical features observed in the data. As in the case of parametric Lorenz curves one can fix suitable parametric PGR for the data. Since any function of t cannot be a PGR, we fix the conditions for an arbitrary function $A_1 : R \to [0, 1]$ to be a PGR.

Theorem 2.2. *A function $A_1 : R \to [0, 1]$ will be the poverty gap ratio relating to a distribution $F(x)$, if and only if it satisfies the following properties*

(a) $\lim_{t\to\infty} \frac{d}{dt} tA_1(t) = 1; \lim_{t\to 0} \frac{d}{dt} tA_1(t) \geq 0$

(b) $A_1(t)$ is increasing and differentiable

(c) $tA_1(t)$ is convex

(d) $A_1(t) = \int_0^t \frac{t-x}{t} dF(x), \quad A_1(t) = A_F(t, 1).$

Example 2.1. Let $A_1(u) = \frac{u^2}{\beta}, \quad 0 \leq u \leq 1, \quad \beta > 1$. Then from (2.6) by direct calculation,

$$Q(u) = \frac{\beta(\beta - 1)}{(\beta - u)^2}.$$

Inverting, X has distribution function

$$F(x) = \beta - \left(\frac{\beta(\beta - 1)}{x}\right)^{\frac{1}{2}}, \quad \frac{\beta - 1}{\beta} \leq x \leq \frac{\beta}{\beta - 1}.$$

The Lorenz curve of this distribution is discussed in Rohde (2009). One can read from Table 1, the distributions specified by quantile functions that are generated by special forms of $A_1(u)$.

Sometimes, the distribution is represented by its quantile density function $q(u)$ and $Q(u)$ may not have a simple enough expression to use (2.4). We observe that

$$A_1(t) = \frac{1}{t} \int_0^t F(x)dx,$$

so that when $t = Q(u)$,

$$A_1(u) = \frac{1}{Q(u)} \int_0^u pq(p)dp. \tag{2.7}$$

A well known case is the distribution specified by

$$q(u) = ku^{\alpha}(1-u)^{\beta}, k > 0,$$

α, β real which contains the exponential, Pareto II, re-scaled beta, log logistic, Govindarajulu distributions as special cases and approximates many continuous distributions. In this case

$$A_1(u) = \frac{B_u(\alpha+2, \beta+1)}{B_u(\alpha+1, \beta+1)}, \alpha, \beta > -2.$$

where $B_u(\alpha, \beta) = \int_0^u p^{\alpha-1}(1-p)^{\beta-1}dp, \alpha, \beta > 0$, is the incomplete beta function. An alternative inversion formula for $Q(u)$ is

$$Q(u) = \exp\left[-\int_u^1 \frac{A_1'(p)}{p - A_1(p)}dp\right]. \tag{2.8}$$

Example 2.2. Assume X to follow power distribution $F(x) = \left(\frac{x}{\alpha}\right)^{\beta}, 0 \leq x \leq 1$ or $Q(u) = \alpha u^{1/\beta}$. Then for this distribution

$$A_1(u) = \frac{u}{\beta+1},$$

proportional to the head count ratio.

A third special case of (2.3) is

$$A_2(t) = A_F(t, 2) = \int_0^t \left(1 - \frac{x}{t}\right)^2 dF(x) \tag{2.9}$$

which measures depth of poverty while $A_F(t, 1)$ indicates its severity. Corresponding to $A_F(t, 2)$, we have

$$A_2(u) = A_Q(u, 2) = \int_0^u \left(1 - \frac{Q(p)}{Q(u)}\right)^2 dp. \tag{2.10}$$

Theorem 2.3. *The function $A_2(t)$ can be determined from $A_1(t)$ as*

$$A_2(t) = \frac{2}{t^2}\int_0^x xA_1(x)dx. \tag{2.11}$$

Example 2.3. For the power distribution $Q(u) = \alpha u^{1/\beta}$,

$$A_2(u) = \int_0^u \left(1 - \frac{p^{\frac{1}{\beta}}}{u^{\frac{1}{\beta}}}\right)^2 dp = \frac{2u}{(\beta+1)(\beta+2)}.$$

Also observe that

$$A_2(u) = \frac{2}{\beta+2}A_1(u).$$

Remark 2.1. We see that $A_2(u)$ and $A_1(u)$ are proportional. From (2.11) it is not difficult to show that $A_2(u) = CA_1(u)$, when C is a constant only X has power distribution. These two measures rank in the same way all power distributions that have the same values for β as observed by one of the referees. This is an interesting result, because if such a property is observed in the data, we can come to the conclusion that the income distribution is power. In that sense the property $A_2(u) = CA_1(u)$ gives a method to identify if the distribution is power.

A somewhat similar poverty index was proposed by Clark et al. (1981) as

$$C_\beta(t,\beta) = \frac{1}{\beta}\int_0^t \left[1-\left(\frac{x}{t}\right)^\beta\right] dF(x), \beta < 1. \tag{2.12}$$

For $\beta = 0$, it reduces to the Watt's measure to be discussed below and for $\beta = 1$, it is the poverty gap ratio. The index (2.12) is the Chakravarty (1983) metric when $\beta > 0$. We have the quantile equivalent of (2.12) as $C_\beta(u,\beta) = \frac{1}{\beta}\int_0^u \left[1-\left(\frac{Q(p)}{Q(u)}\right)^\beta\right] dp$. When $\beta = 2$,

$$C_2(u) = C_2(u,2) = \frac{1}{2}\int_0^u \left(1-\left(\frac{Q(p)}{Q(u)}\right)^2\right) du.$$

Theorem 2.4. *The function $C_2(u)$ is determined form $C_1(u) = A_1(u)$*

When β tends to zero we have the Watt's measure (Watts, 1969)

$$C_0(t,0) = W_F(t) = \int_0^t (\log z - \log x) dF(x)$$

or equivalently

$$\begin{aligned} W_Q(u) &= \int_0^u [\log Q(u) - \log Q(p)]dp \\ &= u \log Q(u) - \int_0^u \log Q(p) dp. \end{aligned} \tag{2.13}$$

Like other measures $W_Q(u)$ also determines the income distribution by the inversion formula

$$Q(u) = \exp\left[-\int_u^1 \frac{W'(p)}{p} dp\right]. \tag{2.14}$$

An example of income model with interesting properties is the power distribution.

Theorem 2.5. *The Watt's measure $W_Q(u) = \frac{u}{\beta}$, if and only if X has power distribution $Q(u) = \alpha u^{\frac{1}{\beta}}$.*

Note that in this case $W(t)$ is proportional to the head count ratio.

Since poverty is to a large extent influenced by the inequalities of income in the population, it is natural that measures of inequality are linked to poverty indices. A major tool in the evaluation of income inequality in the Lorenz curve (LC)

$$L(u) = \frac{1}{\mu} \int_0^u Q(p)dp \tag{2.15}$$

where $\mu = E(X)$ is assumed to be finite. It follows that $L(u)$ determines $Q(u)$ up bo a scale transformation, since

$$Q(u) = \mu \frac{dL(u)}{du}.$$

From (2.15) and (2.5)

$$A_1(u) = u - \frac{L(u)}{L'(u)}.$$

Finally, the Gini index is

$$\begin{aligned} G &= 1 - 2 \int_0^1 L(p)dp \\ &= 1 - 2 \int_0^1 \left(\exp - \int_u^1 (p - A(p))^{-1} dp \right) du, \end{aligned}$$

in terms of PGR.

Generally, the LC is estimated by interpolation or directly from the distribution function of X or through parametric forms. A large number of parametric forms of LC have been proposed in literature. Some important forms, their PGR's and their quantile functions are exhibited in Table 1 for easy reference. As a weakness of parametric LC's it is pointed out that they provide distribution functions that are difficult to manipulate algebraically to extract distributional properties. However a glance at the above table shows that the quantile functions have quite simple forms that admit analytic treatment.

3. Rank-based poverty measures

In continuation of the previous section, here we address a second category of poverty measures from the perspective of quantile functions. The motivation in doing so is the same as that of Section 2, to provide alternative methodology and new results that cannot be obtained by the existing conventional methods.

A second category of poverty metrics called rank-based measures that use a poor person's rank within the poor or whole population as an indicator of relative deprivation have the structural form

$$P(F,t) = \int_0^t q(F,x;z)dF(x). \tag{3.1}$$

Before discussing them, we need some basic concepts that are constituents of (3.1). The first measure is the income gap ratio (IGR) for the poor defined as

Lorenz curve	PGR	$Q(u)$
$u\exp[-\delta(1-u)]$ Kakwani and Podder (1973)	$\frac{u^2\delta}{1+\delta u}$	$\mu(u\delta+1)e^{-\delta}(1-u)$
$u(1-\delta(1-u)^{\frac{1}{2}}), 0<\delta<1$ Kakwani (1980)	$\frac{u^2}{(1-u)^{\frac{1}{2}}+u\delta-\delta}$	$\mu[1-\delta(1-u)^{\frac{1}{2}}+u\delta(1-u)^{-\frac{1}{2}}]$
$\frac{e^{ku}-1}{e^k-1}$ Chotikapanich (1993)	$u-\frac{1}{k}(1-e^{-ku})$	$\frac{\mu e^{ku}}{e^k-1}$
$\frac{(1-\theta)^2u}{(1+\theta)^2-4u}$ Aggarwal (1984)	$\frac{4\theta u^2}{(1+\theta)^2}$	$\frac{\mu(1-\theta^2)^2}{((1+\theta)^2-4\theta u)^2}$
$uT^{u-1}, T>1$ Gupta (1984)	$\frac{u^2\log T}{1+u\log T}$	$\mu[1+\mu\log T]T^{u-1}$
$u[1-(1-u)^\theta], 0<\theta\le 1$ Ortega et al. (1991)	$\frac{\theta u^2(1-u)}{1+u\theta(1-u)^{\theta-1}-(1-u)^\theta}$	$\mu[1+(1-u)^{\theta-1}((\theta+1)u-1)]$
$\frac{u(\beta-1)}{\beta-u}, \beta>1$ Rohde (2009)	$\frac{u^2}{\beta}$	$\frac{\mu\beta(\beta-1)}{(\beta-u)^2}$

Table 1: Functions associated with parametric Lorenz curves

(Sen (1986))

$$I\left(F(t)\right)=1-\frac{\int_0^t xdF(x)}{tF(t)}$$

or equivalently,

$$I_Q(u)=1-\frac{\int_0^u Q(p)dp}{Q(u)}=\frac{1}{uQ(u)}\int_0^u pq(p)dp. \tag{3.2}$$

It gives the average shortfall of incomes among the poor community. We also need the average income of the poor (Nair et al., 2008)

$$\mu_F(t)=\frac{1}{F(t)}\int_0^t xdF(x)=E(X|X\le t)$$

and its quantile form

$$\mu_Q(u)=\frac{1}{u}\int_0^u Q(p)dp=\frac{1}{u}\int_0^u pq(p)dp \tag{3.3}$$

and the Gini index of the poor

$$G_F(t)=1-\frac{2}{\mu_F(t)}\int_0^t x\left(1-\frac{F(x)}{F(t)}\right)\frac{f(x)}{F(t)}dx$$

or

$$G_Q(u)=1-\frac{2}{u^2\mu_Q(u)}\int_0^u (u-p)\,Q(p)dp$$

$$= \frac{2}{u^2 \mu_Q(u)} \int_0^u pQ(p)dp - 1. \tag{3.4}$$

Some of the properties of the functions I_Q, μ_Q and G_Q discussed in Nair et al. (2008) and Nair and Vineshkumar (2022) that are needed in the sequel are the following

(a) the function $D(u)$ determines the income distribution as

$$Q(u) = \frac{\mu}{D(u)} \exp\left[-\int_u^1 D(p)dp\right]$$

where

$$D(u) = u(1 - I_Q(u)).$$

For the power distribution $Q(u) = \alpha u^{\frac{1}{\beta}}$, $I_Q(u) = (1+\beta)^{-1}$, a constant.

(b) $G_Q(u)$ is independent of the poverty quantile u if and only if X has power distribution, where $G_Q(u) = (1 + 2\beta)^{-1}$.

(c) The above measures satisfy the identity with Lorenz curve

(i) $L(u) = \mu^{-1} u \mu_Q(u)$

(ii) $L(u) = \mu^{-1} D(u) Q(u), I_Q(u) = 1 - \frac{\mu L(u)}{uQ(u)}$

(iii) $G_Q(u) = 1 - \frac{2}{uL(u)} \int_0^u L(p)dp$ and $L(u) = \frac{2}{u(1-G-Q)} \exp\left[-\int_0^u \frac{2dp}{p(1-G_Q(p))}\right]$.

It is often difficult to find the distribution function $F(t)$ from $G_F(t)$. An advantage of the quantile approach initiated here is that the distribution $Q(u)$ can be expressed in terms of $G_Q(u)$.

Theorem 3.1. *The income distribution specified by $Q(u)$ is given by*

$$Q(u) = B(u) \exp\left[-\int_u^1 B(p)dp\right] \tag{3.5}$$

where $B(u) = \frac{1+G_Q(p)+pG'_Q(p)}{p\left(G_Q(p)-1\right)}$.

Corollary 3.1. *All the basic functions associated with poverty can be expressed in terms of the PGR, $A_1(u)$.*

(i) the IGR, $I(u) = \frac{A_1(u)}{u}$

(ii) mean income of the poor, $\mu_Q(u) = \frac{1-A_1(u)}{u} Q(u)$

(iii) Lorenz curve $L(u) = \mu^{-1} u \left(u - A_1(u)\right)$

(iv) Gini index of the same

$$G_Q(u) = \frac{\int_0^u pD(p) \exp\left[-\int_p^1 D(r)dr\right]dp}{\int_0^u D(p) \exp\left[-\int_p^1 D(r)dr\right]dp}.$$

The basic rank-based poverty measure proposed by Sen (1976) is

$$S_F(t) = H(t)\left[I_F(t) + (1 - I_F(t))\, G_F(t)\right]$$

where $H(t) = F(t)$ is the head count ratio. We can rewrite it as

$$S_Q(u) = S_F(Q(u)) = u\left[I_Q(u) + (1 - I_Q(u))\, G_Q(u)\right]. \tag{3.6}$$

Since the basic formula (3.6) needs calculation of several constituents, it is useful to have a single expression for $S_Q(u)$ in terms of $Q(u)$. Substituting for I_Q,

$$S_Q(u) = u\left[1 - \frac{\int_0^u Q(p)dp}{uQ(u)} + \frac{\int_0^u Q(p)dp}{uQ(u)} G_Q(u)\right]$$

giving

$$\begin{aligned} u - S_Q(u) &= \frac{\int_0^u Q(p)dp}{Q(u)}\left[1 - G_Q(u)\right] \\ &= 2\frac{\int_0^u Q(p)dp}{Q(u)}\left[1 - \frac{1}{u\mu_Q^{(u)}}\int_0^u pQ(p)dp\right] \\ &= 2\frac{\int_0^u Q(p)dp}{Q(u)}\left[1 - \frac{\int_0^u pQ(p)dp}{u\int_0^u Q(p)dp}\right] \end{aligned}$$

Thus

$$(u - S_Q(u))\, uQ(u) = 2u\int_0^u Q(p)dp - \int_0^u pQ(p)dp$$

leading to the formula

$$S_Q(u) = u - \frac{2\int_0^u Q(p)dp}{Q(u)} + \frac{2\int_0^u pQ(p)dp}{uQ(u)}. \tag{3.7}$$

Conversely,

$$\frac{d}{du} u\,(u - S_Q)\, Q(u) = 2\int_0^u Q(p)dp$$

giving a second order differential equation with variable coefficients

$$u\,(u - S_Q)\frac{d^2y}{du^2} + \left(2u - S_Q - uS_Q'\right)\frac{dy}{du} - 2y = 0 \tag{3.8}$$

with $y = y(u) = \int_0^u Q(p)dp$. It seems that a general expression for $Q(u)$ in terms of $S_Q(u)$is difficult to find. However it is useful in the case of some specific distributions. For example when $S(u) = Ku$, where K is a positive constant (3.8) gives $Q(u) = \alpha u^{1/\beta}$, the power distribution. There are two solutions of (3.8) of which one is the power distribution and the other is a term that cannot be a quantile function so that $S(u) = Ku$ characterizes the power distribution. Some distributions and their expressions for $S(u)$ are given below

(i) power distribution, $Q(u) = \alpha u^{1/\beta}$: $S(u) = \frac{3\beta+1}{(\beta+1)(2\beta+1)}u$

(ii) exponential, $Q(u) = -\frac{1}{\lambda}\log(1-u)$:

$$S(u) = 2 - \frac{1}{u} - \frac{3u-2}{2\log(1-u)}$$

(iii) Pareto I: $Q(u) = \sigma(1-u)^{-\frac{1}{\alpha}}$; $F(x) = 1 - \left(\frac{\sigma}{x}\right)^{\alpha}, x > \sigma > 0$

$$S(u) = u + \frac{2\alpha}{1-\alpha}(1-u)^{\frac{1}{\alpha}} + \frac{2\alpha^2}{(1-\alpha)(1-2\alpha)}u\left[(1-u)^2 - (1-u)^{\frac{1}{\alpha}}\right]$$

(iv) Govindarajuln: $Q(u) = \sigma\left[(\beta_1)\, u^{\beta}\beta_u - \beta u^{\beta+1}\right]$.

$$S(u) = u - \frac{2u}{\beta+1} + 2u^2\left(\frac{\beta+2)(\beta+3) - \beta(\beta+2)u}{(\beta+2)(\beta+3) - (\beta+1-\beta u)}\right.$$

(v) Dagum $Q(u) = b\left[u^{-\frac{1}{\beta}} - 1\right]^{-\frac{1}{a}}$; $F(x) = \left[\left(1 + \left(\frac{b}{x}\right)^{a}\right]^{-\beta}\right.$

$$S(u) = u - \frac{2b}{aQ(u)}\left[B_{\left(\frac{1}{b}\right)^a}\left(\beta + \frac{1}{a}, 1 - \frac{1}{a}\right) - \frac{1}{u}B_{\left(\frac{1}{b}\right)^a}\left(2\beta + \frac{1}{a}, 1 - \frac{1}{a}\right)\right.$$

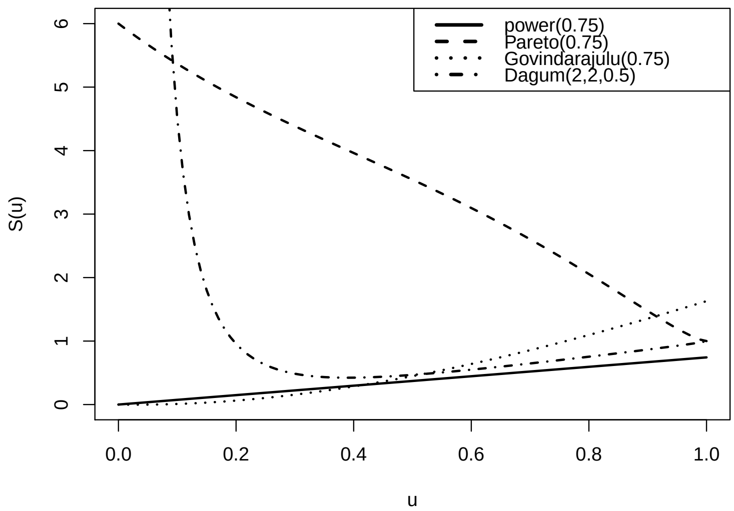

Figure 1: $S(u)$ of power, Pareto, Govindarajulu and Dagum distribution with different parameter values.

In Figure 1 the graphs of the index for the power distribution, Pareto I, Govindarajulu and Dagum distributions are shown. Based on the graph and other analysis, we see that

(i) The index of the power distribution is proportional to the head count ratio and it is the only distribution possessing this property

(ii) $S(u)$ can be increasing (power), decreasing (Pareto I) or initially decreasing and then increasing (Dagum) in u (head count ratio). Since $u = F(t)$ is increasing (decreasing) means t is also increasing (decreasing) and vice-versa the monotonicity of $S_Q(u)$ and $S_F(t)$ are the same.

(iii) When an income model is chosen arbitrarily from a list of candidates to study poverty using $S_Q(u)$, it should be compatible with the monotonic nature of the Sen index.

While Sen index expressed in the general form (3.1) has $q(F,x;z) = 2\left[1 - \frac{F(x)}{r(z)}\right]\left(1 - \frac{x}{z}\right)$, where r_z is the proportion below the poverty line, other prominent indices proposed in literature are

(a) Takayama (1979) with $q(F,x;z) = 2(1-F(x))\left(1 - \frac{x}{\mu_F}\right)$ where $\mu_F = r_z\mu_p + (1-r_z)\,z$ and μ_p is the mean income of the poor.

(b) Kakwani (1980): $q(F,x;z) = (k+1)\,(1 - F(x)/r_z)^k\left(1 - \frac{x}{z}\right)$ and

(c) Thon (1983): $q(F,x;z) = \frac{2}{c-1}\,[c - zF(x)]\left(1 - \frac{x}{z}\right), c \geq 2$

(d) Foster et al. (1984): $F(s) = H\left(I^2(t) + (1-I)^2 C_p^2(t)\right]$ where H is the head count ratio and C_p is the coefficient of variation among the poor. These can also be treated similar to $S(u)$.

Shorrocks (1995) provided a modification of the Sen index which satisfies the properties of symmetry, replication invariance, monotonic and homogeneous of degree zero in incomes and poverty line. See Hagenaars (2017), Xu (2020) and Chakravarty (2019) for details. The Shorrocks-Sen index, as it is called, takes the form

$$SS_F(x) = \mu(x)[1 + G(x)]$$

where μ is the mean and G is the Gini index of $X = \frac{t-Y}{t}$ and Y is the income. In terms of $F(t), I_F(t)$ and $G_F(t)$ we have,

$$SS_F(t) = (2 - F(t))F(t)I_F(t) + F^2(t)\,(1 - I_F(t))\,G_F(t).$$

It is easy to see that the quantile version is

$$SS_Q(u) = (2-u)uI_Q(u) + u^2\,(1 - I_Q(u))\,G_Q(u). \tag{3.9}$$

The form gives properties different from the Sen poverty index, for example, In the power distribution, $SS_Q(u)$ is not proportional to the head count ratio, but

$$SS(u) = \frac{u(2 + 4\beta - (\beta+1)u)}{(\beta+1)(2\beta+1)}$$

is a quadratic function. The derivative of $SS(u)$ is

$$\frac{dSS}{du} = \frac{(2+4\beta) - 2u(\beta+1)}{(\beta+1)(2\beta+1)} > 0 \text{ for all } u.$$

In the interval $[0,1], SS(u)$ is always an increasing function of u.

A third approach to poverty measure by Hagenaars (1987) visualizes it in the tradition of Dalton with the aid of utility functions with the general definition

$$P_F(t) = A(t)\int_0^t [U(t) - U(x)]dF(x)$$

where $U(.)$ is the individual's utility function and $A(t)$ is a normalization factor. This is equivalent to

$$P_Q(u) = P_F(Q(u)) = A_H(u)\int_0^u [g(u) - g(p)]\, dp$$

where $A_H(u) = A(Q(u))$ and $g(p) = U(Q(p))$. This class has some common measures like those of Watts (1969), Chakravarty (1983) and Clark et al. (1981) disused earlier as particular cases.

4. Income modelling by quantile functions

The statistical regularities observed in income date have prompted researchers even from the early days to promote parametric models to represent them. In many cases the data generating mechanism was satisfactorily approximated by distribution functions and the summary measures derived from them were quite useful in indicating income characteristics. These summary measures were often moments, in poverty measures which are partial moments, or dispersion measures like Gini index or mean difference and transformations like Lorenz or Bonferroni curves. However, the absence of any concrete rationale for the choice of a specific model resulted in prescribing a large number of distributions as income models for data separated by time and space. A few important distributions specified by their quantile functions that have potential as income models are given below along with the expressions of poverty measures pertaining to them.

(i) Kappa distribution introduced by Hosking (1994) has quantile function

$$Q_k(u) = \alpha + \frac{\beta}{\gamma}\left[1 - \left(\frac{1-u^\delta}{\delta}\right)^\gamma\right] \tag{4.1}$$

where α is a location parameter and β is a scale parameter. Since the random variable X has to be non-negative, $Q(0) \geq 0$, giving $\alpha + \frac{\beta}{\gamma}(1-\delta^{-\gamma}) \geq 0$ along with $\beta > 0$. The advantage of (4.1) is that it contains as special cases, the Pareto distribution for $\delta = 1$, Stoppa for $\gamma < 0, \delta > 0$ and $\beta = -\gamma(\delta)^\gamma, \alpha = \frac{\beta}{\gamma}$; Dagum, when $\gamma < 0, \delta < 0$, $\beta = \gamma(\delta)^\gamma$ and $\alpha = \frac{\beta}{\gamma}$, Burr type III ($\gamma < 0, \delta < 0$), Burr Type XII ($\gamma > 0, \delta < 0$), generalized Pareto ($\delta = 1$), exponential $\gamma = 1, \delta = 0$. It is also obtained as the distribution of $\frac{1}{Y}$, where Y is the generalized Weibull of Mudholkar and Kollia (1994). Besides the distribution (4.1) gives good approximation to other unimodal distributions $\gamma, \delta < 1$. Tarsitano et al. (2006) has used (4.1) to analyse the income data related to the net disposable in Italian household budget for the years 1993, 1995, 1998 and 2000. Various expressions for poverty measures are

poverty gap ratio, $A_1(u) = u - \frac{\beta P_1(u;\gamma,\delta)}{\gamma\delta Q(u)} - \frac{\alpha u}{Q(u)}$

Lorenz curve, $L(u) = \frac{1}{\mu}\left[\alpha u + \frac{\beta}{\gamma\delta}P_1(u,\gamma,\delta)\right]$,

mean income, $\mu_{Q_k}(u) = \frac{1}{u}\left[\alpha u + \frac{\beta}{\gamma\delta}P_1(u,\gamma,\delta)\right]$,

income gap rato, $I_{Q_k}(u) = 1^{r\delta}\frac{1}{uQ(u)}\left[\alpha u + \frac{\beta}{\gamma\delta}P_1(u,r,\delta)\right]$, and the

Gini coefficient of the poor

$$G_{Q_k}(u) = \frac{2}{u^2\mu_Q}\left[\frac{\alpha u^2}{2} + \frac{\beta}{\gamma\delta}P_2(u,\gamma,\delta)\right],$$

where,

$$P_1(u,\gamma,\delta) = \int (1-t)^{\gamma}(1-t\delta)^{\frac{1}{\delta}-1}dt$$
$$P_2(u,\gamma,\delta) = \int (1-t)^{\gamma}(1-t\delta)^{\frac{1}{s}-1}$$

with the integrals evaluated over $\left(\frac{1-u^{\delta}}{\delta}, \frac{1}{\delta}\right)$.

(ii) Generalized lambda distributions: The form proposed by Ramberg and Schmeiser (1972) has quantile function

$$Q_R(u) = \lambda_1 + \lambda_2^{-1}\left[u^{\lambda_3} - (1-u)^{\lambda_4}\right] \tag{4.2}$$

where $\lambda_2 > 0$ and $\lambda_1 - \frac{1}{\lambda_2} \geq 0$ for non-negativity. The general properties of the model and regions of validity together with conditions that approximate (4.2) to many continuous distribution functions used for income analysis are given in Karian and Dudewicz (2000). Tarsitano et al. (2004) has employed this distribution as an income model. For this model, we have

$$A_1(u) = u - \frac{1}{Q_R(u)}\lambda_1 + \lambda_2^{-1}\left(\frac{u^{\lambda_3+1}}{\lambda_3+1} + P_3(u)\right),$$

$$L(u) = \frac{1}{\mu}\left(\lambda_1 u + P_3(u)\right),\ \mu = \lambda_1 + \lambda_2^{-1}\left(\frac{1}{\lambda_3+1} - \frac{1}{\lambda_4+1}\right),$$

$$\mu_{Q_R}(u) = \frac{1}{\mu}\left(\lambda_1 u + P_3(u)\right),$$

$$I_{Q_R}(u) = 1 - \frac{1}{uQ_R(u)}\left(\lambda_1 u + P_3(u)\right),$$

where $P_3(u) = \frac{u^{\lambda_3+1}}{\lambda_3+1} + \frac{(1-u)^{\lambda_4+1}}{\lambda_4+1}$ and

$$G_{Q_R}(u) = u^2 \frac{2}{\mu_Q(u)} \left[\frac{\lambda_1 u^2}{2} + \frac{u^{\lambda_3+2}}{\lambda_2(\lambda_3+2)} + \frac{u(1-u)^{\lambda_4+1}}{\lambda_2(\lambda_4+1)} + \frac{(1-u)^{\lambda_4+1}-1}{\lambda_2(\lambda_4+2)} \right].$$

An alternative version proposed by Freimer et al. (1988) is

$$Q_F = \lambda_1 + \lambda_2^{-1} \left(\frac{u^{\lambda_3}-1}{\lambda_3} - \frac{(1-u)^{\lambda_4}-1}{\lambda_4} \right)$$

where $\lambda_1 - \frac{1}{\lambda_2\lambda_3} \geq 0$. Haridas et al. (2008) have discussed the role of this distribution for modelling income data. It contains the power, Pareto I & II distributions and approximates the Weibull, Singh-Maddala, Dagum and Fisk distributions for different ranges of parameter values. We have

$$A_1(u) = u - \frac{1}{Q(u)} P_4(u)$$

$$L(u) = \frac{1}{\mu} \left(P_4(u) \right),$$

$$\mu_{Q_F}(u) = \frac{1}{\mu} \left(\frac{1}{\lambda_2\lambda_4 + \lambda_1} + P_4(u) + \frac{(1-u)^{\lambda_{4+1}-1}}{\lambda_2\lambda_4(\lambda_4+1)} \right),$$

$$I_{Q_F}(u) = 1 - \frac{1}{uQ(u)} \left(\frac{1}{\lambda_2\lambda_4} + P_4(u) + \frac{(1-u)^{\lambda_{4+1}-1}}{\lambda_2\lambda_4(\lambda_4+1)} \right),$$

where $P_4(u) = \frac{1}{\lambda_2\lambda_4} + \lambda_1 + \frac{u}{\lambda_2\lambda_4} + \frac{u^{\lambda_3+1}}{\lambda_2\lambda_3(\lambda_4+1)} + \frac{(1-u)^{\lambda_4+1}-1}{\lambda_2\lambda_4(\lambda_3+1)}$ and

$$G_{Q_F}(u) = \frac{2}{u^2\mu_Q} \left[\left(\lambda_1 - \frac{1}{\lambda_2\lambda_3} + \frac{1}{\lambda_2\lambda_4} \right) \frac{u^2}{2} + \frac{u^{\lambda_3+2}}{\lambda_2\lambda_3(\lambda_3+2)} + \frac{u(1-u)^{\lambda_4+1}}{\lambda_4+1} + \frac{(1-u)^{\lambda_4+1}-1}{(\lambda_4+1)(\lambda_4+2)} \right].$$

(iii) Wakeby distribution: Introduced by Houghton (1978), the quantile function is given by

$$Q_w(u) = \lambda + \theta(1-u)^{-\alpha} - \phi(1-u)^{\beta} \tag{4.3}$$

which has five parameters of which λ specifies location, θ and ϕ the scales and α and β the shapes. For the random variable to be non-negative we should have $\lambda + \theta - \phi \geq 0$ and $\theta, \phi > 0$. An important property of the model is that it can satisfactory represent positively skewed data like the log-normal, gamma and beta laws. It contains the Pareto I and Pareto II distributions as special cases. The poverty measures are

$$A_1(u) = u - \frac{1}{Q_w(u)} \quad P_5(u),$$
$$L(u) = \frac{1}{\mu} P_5(u),$$
$$\mu_{Q_W}(u) = \tfrac{1}{u} P_5(u),$$
$$I_{Q_W} = 1 - \tfrac{P_5(u)}{uQ_w(u)},$$

where $\quad P_5(u) = \lambda u + \frac{\theta(1-u)^{-\alpha+1}-1}{\alpha-1} + \phi\frac{(1-u)^{\beta+1}-1}{\beta+1}, \alpha > 1, \beta > -1$ and

$$G_Q(u) = \frac{2}{u^2 \mu_{Q_w}(u)} \left[\frac{\lambda u^2}{2} + \frac{\theta}{\alpha - 1} \left\{ u(1-u)^{-\alpha+1} + \frac{(1-u)^{-\alpha+2} - 1}{\alpha - 2} \right\} + \frac{\phi}{\beta + 1} \left\{ u(1-u)^{\beta+1} + \frac{(1-u)^{\beta+2} - 1}{\beta + 2} \right\} \right].$$

(iv) Govindarajulu distribution: This distribution is defined by

$$Q(u) = \sigma \left((\beta + 1)u^\beta - \beta u^{\beta+1} \right), \sigma, \beta > 0.$$

Its properties and application are described in Nair et al. (2013). In this case

$$A_1(u) = \frac{\beta(\beta + 2 - \beta u - u)}{(\beta + 2)(\beta + 1 - \beta u)},$$
$$L(u) = \frac{1}{2}(\beta + 2 - \beta u)u^{\beta+1},$$
$$\mu_Q(u) = \sigma \frac{(\beta + 2 - \beta u)}{\beta + 2} u^\beta,$$
$$I_Q(u) = \frac{\beta}{\beta + 1 - \beta u},$$

and

$$G_Q(u) = \frac{2(\beta + 2)}{\beta + 2 - \beta_2 u} \left(\frac{\beta + 1}{\beta + 2} - \frac{\beta u}{\beta + 3} \right).$$

Other flexible quantile functions that have a similar role to play in income analysis can be seen in Nair et al. (2013).

Remark 4.1. All the distributions except (i) given above do not have tractable distribution functions and hence the conventional methods cannot be applied to them.

Remark 4.2. In the above formulas setting $u = 1$ in $\mu_Q(u)$ gives the mean of the distribution and similarly $u = 1$ in $G_Q(u)$ leads to the usual Gini index.

5. Applications to data analysis

In this section we illustrate the role of quantile functions as models of income and how the poverty measures are computed from them for real data. Often the income data is given for a sample in the population and therefore standard inferential procedures are required for the computation of the measures for the population. This is often accomplished by identifying the income distribution from the sample data. It serves the twin purposes of determining the law that has generated the observations and also to obtain estimates of the poverty measures that is not affected by sampling fluctuations.

The observations pertain to the disposable household income of Austria for the year 2022, obtained in the survey on income and living conditions carried out in the survey year 2023 by Statistics Austria. The data is obtained from Luxembourg Income Study (LIS) (2022). The data set consisting of 6122 observations cover households living in accommodations with atleast one member whose main residence is in the household at the time of data collection. On record are the total income available to a household after taxes and social contributions.

The candidate distribution selected to model the data is the power-Pareto distribution with quantile function

$$Q(u) = C\frac{u^{\lambda_1}}{(1-u)_2^{\lambda}}, \tag{5.1}$$

where $C > 0$, and λ_1 and λ_2 are real numbers. This distribution was constructed using the result that the product of two positive quantile functions is again a quantile function, by choosing the product of the quantile functions of the power distribution $Q_P(u) = C_1 u^{\lambda_1}$ and the Pareto I quantile function $Q_X(u) = C_2(1-u)^{-\lambda_2}$. See Gilchrist (2000) for details. The power distribution capable of modelling incomes in the left tail, Pareto for the right tail are members of (5.1), besides providing close approximations to positively skewed forms of the lognormal, gamma, Weibull and Dagum distributions. The detailed properties of the distributions are available in Hankin and Lee (2006) and Nair et al. (2013).

To estimate the parameters of (5.1) we have chosen the method of selected points by equating the quartiles $Q(0.25)$, $Q(0.5)$, and $Q(0.75)$ with those of the sample $Q_1 = 27435$, $Q_2 = 44199$, and $Q_3 = 66490$, resulting in the following equations

$$\begin{aligned} C(0.25)^{\lambda_1}(0.75)^{-\lambda_2} &= 27435 \\ C(0.5)^{\lambda_1}(0.5)^{-\lambda_2} &= 44199 \\ C(0.75)^{\lambda_1}(0.25)^{-\lambda_2} &= 66490. \end{aligned}$$

Solving these, we got the estimates as

$$\hat{C} = 52134.5, \quad \hat{\lambda}_1 = 0.521999, \quad \text{and} \quad \hat{\lambda}_2 = 0.283774.$$

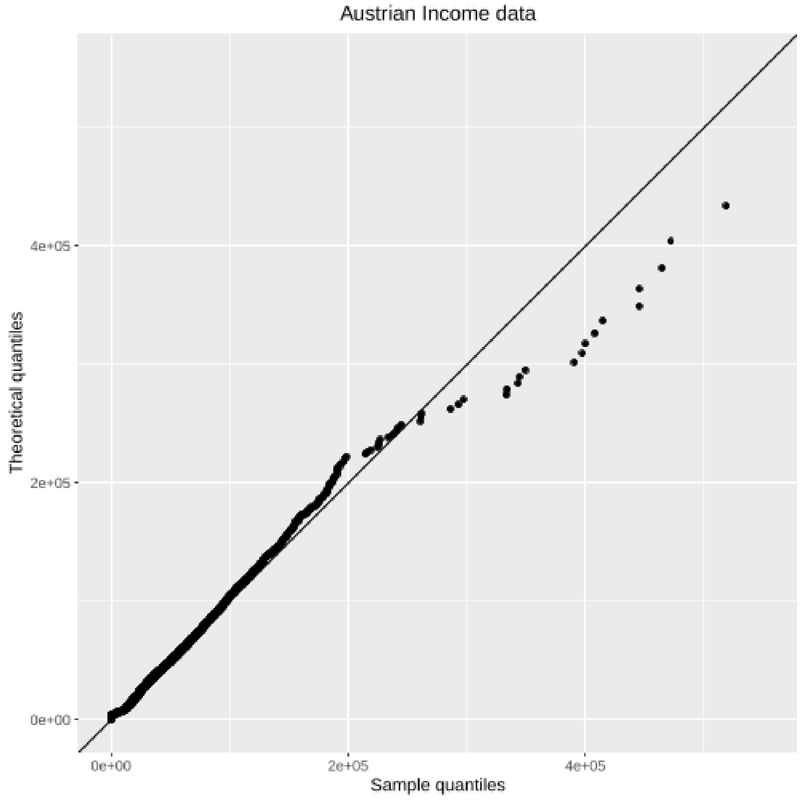

Figure 2: Q-Q plot

The adequacy of the power-Pareto distribution as the model to represent the income was ascertained visually by the Q-Q plot and theoretically by method using Q-Q plot given in Nair et al. (2024). As seen in Figure 2 the points in the Q-Q plot lie almost on a straight line (the departure of a very small percentage of points from the line due to abnormally high incomes, is an inherent feature in most Q-Q plots) indicates that the fit is satisfactory. To confirm this we apply a test of goodness of fit from the Q-Q plot, based on the fact that asymptotically the correlation between the empirical quantiles and the theoretical quantiles is one, if the distribution from the sample and the hypothesised model are identical. Thus for moderate sample a high correlation will indicate the adequacy of the fit. Since the theoretical distribution of the correlation coefficient between the two sets of quantiles is difficult to find, the sampling distribution is obtained by simulation of samples of desired size. For our case, critical region is samples whose correlation coefficient is $r_{0.05} = 0.9322$, while the observed value is $r = 0.9919$. Thus the hypothesis of power-Pareto distribution for the Austrian household income data is not rejected. That is the distribution is specified by

$$\hat{Q}(u) = 52134.5\ u^{0.521999}\ (1-u)^{-0.283774}, \quad 0 \le u \le 1. \tag{5.2}$$

There are several advantages the quantile models have in comparison with distribution functions in the traditional approach. Firstly they are flexible as most of them have parameters that control both the left and right tails and hence can fit different types of data structures. When $F(x)$ is continuous and strictly increasing $F(x) = u$ implies $x = Q(u)$ and conversely. Hence the percentile values can be compared easily by taking $u = \frac{i}{100}, i = 1, 2, \cdots, 99$, in (5.2). Thus the incomes at the deciles of the income distributions calculated as

$$x_i = Q(u_i), \quad i = 1, 2, \cdots, 9$$

are exhibited in Table 2. On the other hand if specific head count ratios are re-

Table 2: Incomes at the deciles of the income distribution

$F(x)$	0.10	0.20	0.30	0.40	0.50	0.60	0.70	0.80	0.90
x	16147.71	23975.30	30770.90	37355.61	44198.99	51789.94	60904.02	73263.52	94844.87

quired at desired incomes, then again put the incomes as $Q(u)$ in (5.2) and solve for the corresponding u. Moreover if summary measures of the income distributions are required they are directly obtained from $Q(u)$ as the median $Q(0.5) = 44198.99$ for location, quartile deviation $S = Q(0.75) - Q(0.25) = 39055$ for dispersion, Bowley's coefficient of skewness $\frac{1}{S}\left(Q(0.25) + Q(0.75) - 2Q(0.5)\right) = 0.1415$ and Moore's measure of kurtosis $\frac{1}{S}\left(Q(0.875) - Q(0.625) - Q(0.375) + Q(0.125)\right) = 0.4207$ without evaluating several integrals involving the density function when moments are used in the conventional approach. A further advantage is that the quantile based measures are more stable being less affected by the extreme observations due to very few large incomes than the moment based measures.

The estimates of the mean income of the poor is

$$\begin{aligned}\mu_Q(u) &= \frac{1}{u}\int_0^u C\frac{p^{\lambda_1}}{(1-p)^{\lambda_2}}\, dp \\ &= \frac{C}{u}B_u(\lambda_1 + 1, 1 - \lambda_2) \\ &= \frac{52134.5}{u}B_u(1.521999, 0.716226)\end{aligned}$$

Table 3 gives the values of $\mu_Q(u)$ for the Austrian incomes at $u = 0.05$ through $u = 0.95$.

The Gini index arising from (3.4) is

$$\begin{aligned}G_Q(u) &= \frac{2}{u^2\mu_Q(u)}\int_0^u pQ(p)\, dp - 1 \\ &= \frac{2C}{u^2\mu_Q(u)}B_u(\lambda_1 + 2, 1 - \lambda_2) - 1 \\ &= \frac{104269}{u^2\mu_Q(u)}B_u(2.521999, 0.716226) - 1\end{aligned}$$

Using the estimates of λ_1 and λ_2 and $\mu_Q(u)$ from Table 3, we have the values in Table 4.

Table 3: Mean income of the poor

u	$\mu_Q(u)$
0.05	7233.77
0.10	10481.99
0.15	13075.85
0.20	15345.33
0.25	17420.12
0.30	19368.40
0.35	21232.33
0.40	23041.25
0.45	24817.66
0.50	26580.47
0.55	28347.07
0.60	30134.89
0.65	31963.08
0.70	33854.47
0.75	35838.80
0.80	37958.40
0.85	40280.57
0.90	42929.66
0.95	46205.75

Another quantity of interest is the Lorenz curve which is obtained as

$$L(u) = \frac{u\mu_Q(u)}{\mu},$$

where μ is the mean estimated as $\hat{C}B(\hat{\lambda}_1 + 1, 1 - \hat{\lambda}_2)$, and $\mu_Q(u)$ is chosen from Table 3. The graph of the Lorenz curve is shown in Figure 3.

Finally, we have the Sen index from (3.7) as

$$\begin{aligned}\hat{S}(u) &= u - \frac{2\hat{C}B_u(\hat{\lambda}_1 + 1, 1 - \hat{\lambda}_2)}{u^{\hat{\lambda}_1}(1-u)^{-\hat{\lambda}_2}} + \frac{2\hat{C}B_u(\hat{\lambda}_1 + 1, 1 - \hat{\lambda}_2)}{u^{\hat{\lambda}_1+1}(1-u)^{-\hat{\lambda}_2}} \\ &= u - \frac{104269B_u(1.521999, 0.716226)}{u^{0.521999}(1-u)^{-0.283774}} + \frac{104269B_u(1.521999, 0.716226)}{u^{1.521999}(1-u)^{-0.283774}}.\end{aligned}$$

The graph of the Sen index is given in Figure 4.

6. Conclusion

We conclude this work by proposing quantile functions as models of income distribution, defining and establishing properties of poverty measures derived from them, as alternatives to the traditional ones based on distribution functions. This has resulted some new models and alternative methodologies for poverty analysis, which are more robust and sometimes easier.

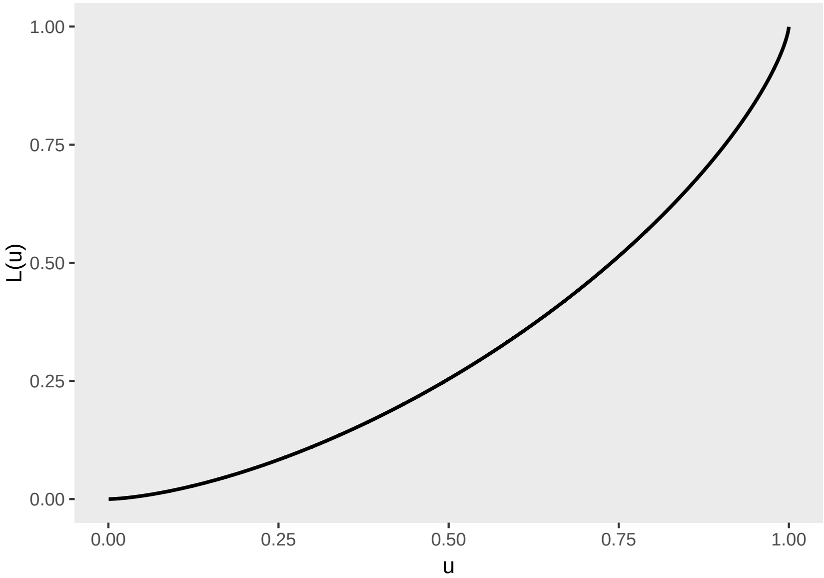

Figure 3: Lorenz curve for the Austrian data

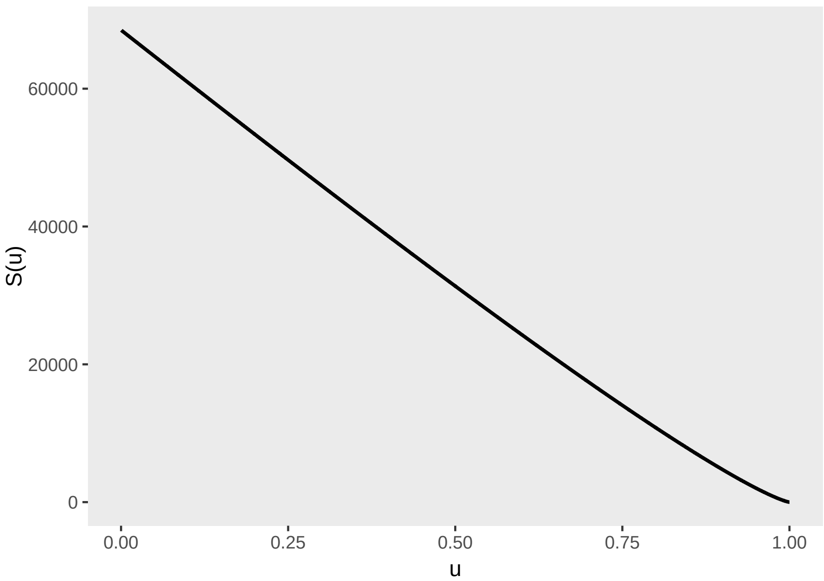

Figure 4: Sen index for the Austrian data

Table 4: Gini index

u	$G_Q(u)$
0.05	1.2090
0.10	1.2111
0.15	1.2133
0.20	1.2157
0.25	1.2182
0.30	1.2209
0.35	1.2238
0.40	1.2270
0.45	1.2304
0.50	1.2342
0.55	1.2383
0.60	1.2429
0.65	1.2480
0.70	1.2539
0.75	1.2607
0.80	1.2688
0.85	1.2788
0.90	1.2918
0.95	1.3107

Data Availability Statements

Authors can confirm that all relevant data are included in the article and/or its supplementary information files.

Conflict of interest statement

On behalf of all authors, the corresponding author states that there is no conflict of interest.

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Appendix A.

Theorem. *The income distribution is uniquely determined by $A(u)$ through the formula*

$$Q(u) = \frac{\exp\left[-\int_u^1 (p - A_1(p))^{-1}\,dp\right]}{u - A_1(u)}. \tag{A.1}$$

Proof. From (2.5),

$$\frac{Q(u)}{\int_0^u Q(p)dp} = (u - A_1(u))^{-1}$$

or

$$\frac{d}{du}\log\int_0^u Q(p)dp = (u - A_1(u))^{-1}$$

giving

$$\int_0^u Q(p)dp = \exp\left[-\int_u^1 (p - A_1(p))^{-1}dp\right].$$

Differentiating the last equation we have (A.1). □

Appendix B.

Theorem. *A function $A_1 : R \to [0,1]$ will be the poverty gap ratio relating to a distribution $F(x)$, if and only if it satisfies the following properties*

(a) $\lim_{t\to\infty} \frac{d}{dt}tA_1(t) = 1; \lim_{t\to 0} \frac{d}{dt}tA_1(t) \geq 0$

(b) $A_1(t)$ is increasing and differentiable

(c) $tA_1(t)$ is convex

(d) $A_1(t) = \int_0^t \frac{t-x}{t}dF(x), \quad A_1(t) = A_F(t,1).$

Proof. Let $A_1(t)$ be as in (d). Then

$$tA_1(t) = \int_0^t (t-x)dF(x) = \int_0^t F(x)dx$$

giving

$$F(t) = \frac{d}{dt}tA_1(t)$$

It is not difficult to see that $F(x)$ satisfies all the conditions of a distribution function of which $A_1(t)$ is the PGR by definition. Conversely if $F(x)$ is the distribution function with PGR; $A_1(t)$, then properties (i) to (iv) hold as seen from the above. □

Remark. *The corresponding properties of $A_1(u)$ can be obtained as in Theorem 2.2. We require $A_1(u)$ to satisfy the conditions that*

i) $Q(u) = \frac{\exp\left[-\int_u^1 (p-A_1(p))^{-1}dp\right]}{u-A_1(u)}$

ii) $A_1(u)$ is an increasing function for $0 < u < 1$ and

iii) $A_1(u)$ is continuous.

Appendix C.

Theorem. *The function $A_2(t)$ can be determined from $A_1(t)$ as*

$$A_2(t) = \frac{2}{t^2}\int_0^x xA_1(x)dx. \tag{C.1}$$

Proof. Since $\frac{1}{t}\int_0^t F(x)dx = A_1(t)$, we have $F(t) = \frac{d}{dt}tA_1(t)$. Hence

$$\begin{aligned} t^2A_2(t) &= \int_0^t (t-x)^2 dF(x) \\ &= 2\int_0^t (t-x)dF(x) \\ &= 2\int_0^t (t-x)\left(\frac{d}{dx}xA_1(x)\right)dx \\ &= 2\int_0^t xA_1(x)dx, \end{aligned}$$

from which (C.1) follows. □

Appendix D.

Theorem. *The function $C_2(u)$ is determined form $C_1(u) = A_1(u)$*

Proof. That $C_1(u) = A_1(u)$ is mentioned above. Now,

$$\begin{aligned} A_2(u) &= \int_0^u \left(1 - \frac{Q(p)}{Q(u)}\right)^2 dp \\ &= u - \frac{2}{Q(u)}\int_0^u Q(p)dp + \frac{1}{Q^2(u)}\int_0^u Q^2(p)dp \end{aligned}$$

giving

$$\begin{aligned} &\frac{1}{Q^2(u)}\int_0^u Q^2(p)dp = A_2(u) + \frac{2}{Q(u)}\int_0^u Q(p)dp - u \\ &= A_2(u) + 2\left(u - A_1(u)\right) - u \end{aligned}$$

Hence

$$\begin{aligned} C_2(u) &= \frac{1}{2Q^2(u)}\left[\int_0^\infty \left[Q^2(u) - Q^2(p)\right]dp\right. \\ &= \frac{u}{2} - \frac{1}{2Q^2(u)}\int_0^u Q^2(p)dp \end{aligned} \tag{D.1}$$

From (2.12) and (D.1)

$$C_2(u) = A_1(u) - \frac{1}{2}A_2(u)$$

Using Theorem 2.3, $A_2(u)$ is determined from $A_1(u)$ and $A_1(u) = C_1(u)$ and Theorem 2.4 is proved. □

Appendix E.

Theorem. *The income distribution specified by* $Q(u)$ *is given by*

$$Q(u) = B(u) \exp\left[-\int_u^1 B(p)dp \right] \tag{E.1}$$

where $B(u) = \frac{1+G_Q(p)+pG'_Q(p)}{p\big(G_Q(p)-1\big)}$.

Proof. From (3.4),

$$\frac{1-G(u)}{2}\mu_Q(u) = \int_0^u \left(\frac{1}{u} - \frac{p}{u^2}\right) Q(p)dp.$$

Using (3.2),

$$\frac{1-G(u)}{2}\int_0^u Q(p)dp = \int_0^u Q(p)dp - \frac{1}{u}\int_0^u pQ(p)dp$$

which simplifies to

$$\frac{1+G(u)}{2}u\int_0^u Q(p)dp = \int_0^u pQ(p)dp$$

Differentiating with respect to u, we get after some algebra,

$$u[(G(u)+1) - 2u]Q(u) = [uG'(u) + G(u) + 1]\int_0^u Q(p)dp$$

and

$$\frac{Q(u)}{\int_0^u Q(p)dp} = B(u)$$

or

$$\frac{d}{du}\log\left(\int_0^u Q(p)dp\right) = B(u)$$

and

$$\int_0^u Q(p)dp = \exp\left[-\int_u^1 B(p)dp \right] \tag{E.2}$$

The proof is completed by noting that (E.1) is the derivative of the last expression (E.2). □