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The Intensity and Shape of Inequality: The ABG Method of Distributional Analysis

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The Intensity and Shape of Inequality: The ABG Method of Distributional Analysis¹

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Abstract

Inequality is anisotropic: its intensity varies by income level. We here develop a new tool, the isograph, to focus on local inequality and illustrate these variations. This method yields three coefficients which summarize the shape of inequality: a main coefficient, α , which measures inequality at the median, and two correction coefficients, β and γ , which pick up any differential curvature at the top and bottom of the distribution. The analysis of a set of 232 microdata samples from 41 different countries in the LIS datacenter archive allows us to provide a systematic overview of the properties of the ABG ($\alpha \beta \gamma$) coefficients, which are compared both to a set of standard indices (Atkinson indices, generalized entropy, Wolfson polarization, etc.) and the GB2 distribution. This method also provides a smoothing tool that reveals the differences in the shape of distributions (the strobiloid) and how these have changed over time.

JEL codes: D31, C16, C46.

Keywords: inequality, distributions, comparisons, polarization, isograph, strobiloid.

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1. Introduction

The analysis of income distribution is central for our understanding of the structure of inequality and social transformations. In his seminal work on distributions, Pareto (1896:99, 1897: v2.305-24) proposed a leptokurtic distribution, which provides a good approximation to the top of the income hierarchy, and provided graphical representations based on incomes (Pareto, 1909: 380-8). Improvements have been made since the introduction of the Gini index (Gini, 1914), but the outdated tools still in use have produced the general conception that inequality is a single-dimensional concept, even though these tools can provide a variety of results.² The current contribution intends to show how local inequality can vary along the income scale. This idea is rooted in the traditional literature on the problem of ranking income distributions (Atkinson and Bourguignon, 1982, Shorrocks, 1983) and dominance issues (Yitzhaki, 1982), and is consistent with the development of inequality indices which are sensitive to specific segments of the distribution (Atkinson, 1970). Our aim here is to distinguish inequality at the middle, top and bottom of the distribution.

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This is a meaningful question for income distributions, as can be shown in an empirical example.³ Table 1 shows the quantiles of the income distribution in Israel in 2010 (il10) and

² There have been obvious improvements in our understanding of the socioeconomic processes which can generate these Pareto distributions (Gabaix, 2009), and even the double Pareto (Reed, 2001) since the lower tail has this particular shape as well. In this field, general surveys (Kleiber and Kotz, 2003) illustrate the diversity of approaches. Over time, more appropriate and more general statistical distributions have been developed, from the Champernowne-I (1937) and Fisk (1961) distributions to the Generalized B of the second kind (GB2) that are becoming standard tools (Jenkins, 2009). In parallel, many inequality indices have been developed (Champernowne and Cowell, 1998:151-3) and a mass of harmonized data has been accumulated (Brandolini and Atkinson, 2001, Cowell 2000, 2003, 2005). In addition, the graphical innovations used to represent distributions have been reviewed by Dombos (1982), who listed dozens of graphical models in addition to the still used log-log Pareto diagram (Nirei and Souma, 2007:444), the Lorenz curve (1905), Pen's Parade (1971), as well as standard density, cumulative distribution function or quantile function graphs. The field of inequality analysis may therefore seem like a mature technology.

³ In this paper, the measurement units are individuals and their income is defined as household disposable (after tax and transfers) cash income per consumption unit (the square root of the number of household members), divided by the median income of the population. Zero or negative points are excluded from the analyses. The

the U.S. in 2010 (us10). The Gini coefficients of both series are similar, at 0.387 and 0.371 respectively.⁴ However, the comparison of the distributions in Table 1 reveals considerable differences. In 2010 in Israel, the fifth percentile level (p5) was 30.1% of the median (p50) and percentile 95 (p95) was 2.95 times the median. Close to the median there was less inequality in the U.S. than in Israel. However, in the lower quantiles, the poorer Israeli residents are relatively better off than their U.S. counterparts by far, and the richest percentile p99 was closer to the median in Israel than in the U.S. Hence, in Israel, there was more inequality around the middle and less inequality at the extremes of the distribution, with this being particularly the case at the bottom. In terms of “general inequality” in 2010, as conventionally reflected in the Gini coefficient, for instance, Israel is slightly more unequal than the U.S. But in terms of “local” inequality, a notion that can be intuitively defined as a local stretching-out of the distribution, the Israel/U.S. comparison is obviously much more complicated, with there being both more and less inequality across various segments of the income distribution. This kind of ambiguous situation is related to the well-known problem of the comparison of Gini coefficients when the associated Lorenz curves cross each other. We aim to resolve this ambiguity by generalizing the idea of diversity in “local inequality” over the income distribution.⁵ We propose an analysis in terms of the shape of inequalities that has in general been neglected to date.⁶

We first discuss how the well-known Champernowne I – Fisk (CF) distribution (Champernowne, 1937, Fisk, 1961) can be used as a baseline for local inequality analysis.

From this baseline, we propose the “Isograph”, a tool which represents the diversity of local

term “medianized equivalized disposable income” (medi) refers to this income concept. The same method can be adapted for the analysis of wealth inequality (Jäntti *et al.*, 2013).

⁴ The country codes are based on the International Organization for Standardization two-character codes (www.iso.org/iso/country_codes) followed by the survey year.

⁵ Gabaix (2009) does consider this local degree of inequality, but his topics (mainly the size of cities, firms, and the largest actors on the stock market) lead to a focus on the top of the distribution and not on the whole scale: with city sizes, α is close to 1 (the Zipf law), and so the description of a “median size city” is somewhat perplexing.

⁶ Weeden and Grusky (2012) recently focused on the forms of inequality but in terms of categorical groupings rather than the distribution of economic resources.

inequality over the income distribution: this reveals how the empirical degree of inequality can be deducted from the CF hypothesis at the median but with additional curvature at the top and bottom of the distribution.⁷ We therefore propose an α, β, γ (ABG) method of estimating three inequality parameters, compatible with the Pareto properties of the tails. The related coefficients are directly interpretable in terms of level-specific measures of inequality: the central coefficient (α) measures inequality at the median level with correction parameters at the top (β) and bottom (γ). An empirical analysis of 232 datasets from 41 different countries provides estimates of the ABG coefficients. The ABG results are compared to 30 conventional and specialized inequality indicators/coefficients, and we also compare its ability to fit empirical distributions with that of the GB2, which can certainly be considered as the most influential distribution in contemporary income analysis (McDonald, 1984, Jenkins 2009). The advantages of this ABG method are its ability to fit empirical cases, to help us understand the shapes of the distributions (strobiloids) and to provide interpretable coefficients.

2. The CF distribution as a baseline

The Champernowne-Fisk distribution is one of the many statistical laws used to model incomes. We cannot claim that the CF is the best curve – the GB2 provides a better fit since it is more flexible with two additional parameters – but it does provide a simple template which is able to pick up changes in local inequality.

In this CF tradition, we can approximate an income distribution as in equation (1). Consider each individual i ($i=1, \dots, n$) with income $y_i > 0$; she is above a proportion p_i of individuals (p_i is the so-called “standardized quantile” pertaining to income level y_i , otherwise called the

⁷ The isograph presents the slope of the “Fisk Graph” (Fisk, 1961:176) that is indeed a logit-log transformation of the Pen’s parade (Pen, 1971:49–59), a transformation of the cumulative distribution function graph. In the Fisk Graph, compared to the early Fisk proposal of 1961, the axes are inverted (like in a quantile function) so that a log income pertains to a logit-percentile position. This improves the traditional Pareto graph.

“fractional rank” (see Jenkins and Van Kerm, 2009). The general quantile distribution expression of the CF of the shape parameter α (CF_α) is particularly simple, provided that we consider medianized incomes (i.e., income divided by the median), $m_i=(y_i/\text{median})$:

$$\ln(m_j) = \alpha \ln(p_i/(1 - p_i)) \quad (1)$$

$$\text{or } M_i = \alpha X_i$$

where $X_i = \text{logit}(p_i) = \ln(p_i/(1 - p_i))$ and $M_i = \ln(m_i) = \ln(y_i/\text{median})$.

Expression (1) is precisely a CF_α , where α measures the degree of inequality understood as the stretching out of the distribution curve.

There are three types of strong arguments which support the use of a CF_α as a first approximation to income distributions.

First, with its two-parameter formula (the median and α), the CF is one of the most parsimonious laws with appropriate Pareto-type power-tails at both extremes, and its formula is remarkably simple. In the CF, log medianized income is proportional to the log-odds of the standardized quantile. This parsimony is notable, and the coefficient $\alpha \in]0,1[$ in the CF_α has a remarkable role in the measurement of inequality since its value is the Gini coefficient.⁸

Second, the CF has a particular position in the field of distributions (McDonald and Xu, 1995: 139). It is central in the general tree of Beta-type distributions (Kleiber and Kotz, 2003: 188) where GB2 is in this sense the canopy of the tree and the CF the roots. The CF is a very

⁸ With his parameterization of the CF cumulative-distribution function $F(k; \lambda; \delta) = (1 + \lambda k^{-\delta})^{-1}$, $k > 0$, $\lambda > 0$, $\delta > 1$, Dagum (2006: 245) demonstrates that $\text{Gini} = \delta^{-1}$, where $\delta > 1$ is the shape parameter of the Fisk distribution. In particular, the α of the ABG is equal to Dagum's δ^{-1} , so $\text{Gini} = \alpha$. This reformulates an earlier publication by Dagum (1975), cited in Kleiber and Kotz (2003: 224), where they use different notation with Dagum's δ scale parameter denoted by a . Thus, in equation (1) here, the parameter α is an inequality coefficient equal to the Gini index, provided that $\alpha < 1$. In the case of a discrete population, α can be greater than 1: an example is the distribution of the number of war casualties over the last century (Cederman, 2003), where α is estimated to be 1.5. In the Zipf distribution (Gabaix, 1999), which is typical of city-size distributions, α is 1. In these cases of discrete distributions with high values of α , the continuous expressions of the mean size produce integrals that diverge to infinity. In this case, the usual Gini formula and the estimation of α can produce divergent results. This is never the case with income distributions, where the highest Gini coefficients are below 0.7.

simplified GB2 where the parameters p and q equal 1. While the CF is much less flexible than the GB2, it does share some important features, such as power-tails. The CF is a sub-case of the complete Champernowne-II (1937, 1952) four-parameter distribution; Fisk (1961) described this simplified form more generally. He called this the “sech² distribution” (the square of the hyperbolic sequant); it is also called the log-logistic distribution (Shoukri *et al.*, 1988, Dagum, 1977, 2006).

Third, the CF produces income distributions that are solidly grounded in mathematical expressions.⁹ Here the CF is at a crossroads of different theoretical traditions. In microeconomics, the GB2 (and, as a consequence, the CF which is a GB2 with parameters $p=q=1$) can be seen as a result of Parker’s neoclassic model of firm behaviour (Parker, 1999:199, Jenkins, 2009).¹⁰ A number of other theoretical constructions, such as stochastic processes of income attainment, yield the same distribution.¹¹ In a proposal from the field of finance, Gabaix (2009) considers stochastic models based on geometric Brownian motion that can generate this type of distribution.

In the social sciences, the balance of power theory of incomes also generates CF laws. This theory assumes proportionality between the power of income and the power of rank.

Developed societies are socially hierarchized on the basis of rank (of education, prestige, political power, or “value” of any kind) which can be expressed as a standardized rank p in $]0,1[$. Each individual i ($i=1, \dots, n$) with income y_i is above a proportion p_i of individuals and has a proportion of $q_i = 1 - p_i$ individuals above him. Since the “power of income”

⁹ Some functional forms “claim attention, not only for their suitability in modeling some features of many empirical income distributions, but also because of their role as equilibrium distributions in economic processes” (Cowell, 2002:25-6).

¹⁰ Using Parker’s parameterization, when α , a constant production elasticity, is set equal to $\frac{1}{2}$, and the elasticity of income returns with respect to human capital γ equals $(b-1)/2$, then $p=q=1$, so that this GB2 is a CF_b.

¹¹ It is still unclear how the CF income distribution is actually related to the stochastic processes developed by Champernowne (1953), a proposition that was reworked by Shorrocks (1975) in his analyses of stochastic models of income attainment, and recently renewed by Reed (2001) and Gabaix (2009). See Kleiber and Kotz (2003, 65sq) also. Osberg (1977) criticized this stream of research on the basis of its *ad hoc* way of mimicking reality, inexact predictions and implicit belief that hierarchy is the result of random processes.

(Champernowne, 1937) is defined as $Y_i = \ln(y_i)$, the “power of social rank” (or “logit rank” X_i) can be defined as the logit of the rank quantile p_i : $X_i = \ln(p_i/q_i) = \text{logit}(p_i)$.¹² Consider two individuals (i) and (j): their difference in power of income $\Delta Y = Y_j - Y_i$ is proportional to the difference in their power of rank, $\Delta X = X_j - X_i$. Then, $\Delta Y = \alpha \Delta X$, where the constant α reflects the intensity of economic inequality in this society. The income inequality between (i) and (j) can thus be derived from the social power of rank:

$$\ln(y_j/y_i) = \alpha \ln\left(\frac{p_j}{1-p_j}\right) \left(\frac{1-p_i}{p_i}\right) \quad (2)$$

The higher is p_i , the greater is the power of social rank; as p_i tends to 1, the power of social rank tends to $+\infty$. This could explain why, at the top of the distribution of prestige, it is strategic to increase rank, as the rewards in terms of $\text{logit}(\text{quantile})$ tend to infinity, and the cost of losing rank is very high, and obviously much larger than that in the neighborhood of the median. Equally, close to the bottom, gaining/losing rank may have immense consequences in terms of the power of rank and relative income. This could explain why Aristotle sees the top of the distribution as dangerously arrogant and the bottom prone to brutality, while the middle of the scale corresponds to stability and moderated political attitudes (Aristotle, 1944:329). One important consequence of equation (1) is the existence of a “sling effect”, since, as α increases, the consequences of a percentage change in income can be significant close to the median but critical at the extremes of the distribution.

In detail, under a CF_α distribution, a change of one percentage point in α generates an increase of income of about one percentage point near the third quartile ($X=0.98$), about two percentage points near the ninth decile ($X=2.197$), about three near the top 95% ($X=2.944$),

¹² Among others, Clementi and colleagues (2012) log-transform the value of rank, even though the quantile, which is an $]0,1[$ interval variable, should be transformed symmetrically (around 0.5) which is what the logit transformation does. Similarly, in the sociology of stratification, Tony Tam (2007) introduced the positional status index (PSI), p_i/q_i that we log here. The concept of “logit rank” is more common in epidemiology than in the social sciences. “Logit rank” (O’Brien, 1978, Copas, 1999), “logistic quantile” (Orsini and Bottai, 2011) or other names for logit rescaling of $]0,1[$ proportions exist in the literature but have not received the attention they deserve.

and so on. As the Gini coefficient rises, extreme top-incomes gain a much higher percentage in terms of their initial income than do the upper middle class. Symmetrically (in terms of log), the poor suffer from greater percentage declines in resources than do the lower middle class.

A number of different fields of research (microeconomics, finance, statistics, and social sciences) thus confirm the importance of the CF, although the adequacy of its description of empirical reality remains to be established. The CF is not the best curve in general: since the GB2 has two additional parameters, it should provide a better fit. Even so, the CF is a parsimonious relevant baseline or template for inequality, playing a central role as a simple equilibrium distribution resulting from economic processes. We can expect that the CF (like the other theoretically-based distributions) will not perfectly fit any type of empirical curve because, in advanced economies, the equilibrium distributions are necessarily distorted at their extremes by social policies, progressive taxation, redistribution, public incentives, and the processes of access to power and their consequences. The CF is thus not the perfect curve but rather a template which is able to detect empirical divergences from theoretical equilibria. Nonetheless, the strong hypothesis here is that, even if it is not the best curve, the CF is empirically relevant in the field of income distribution.

3. Measuring empirical divergences from the CF distribution

The analysis of empirical distributions confirms that expression (1) is a first-order approximation that can be improved upon (Appendix 1: 232 Isographs). I propose the introduction of an ISO function that generalizes (1) into equation (3) and, thereby picks up the divergence of the empirical curve from the CF hypothesis:

$$M_i = \text{ISO}(X_i) X_i, \text{ where } M_i = \ln(y_i/\text{median}) \quad (3)$$

Simply, ISO represents the ratio M/X . If $ISO(X_i)$ is a constant (α), (3) simplifies to (1) and the distribution is a CF_α that equals the Gini index; the higher the value of α , the greater is inequality.

In general, the CF distribution hypothesis does somewhat diverge from reality. Therefore, the isograph representing $ISO(X_i)$ is not a constant and expresses the intensity and the shape of local inequality. The higher is $ISO(X_i)$, the greater the stretching out of incomes at the logit rank level X_i . The change in $ISO(X_i)$ along the distribution measures “local inequality”, which can be thought of as the local stretching of the distribution.

PLACE FIGURE 1 HERE

The empirical isographs are horizontal lines that are often bent at the two extremes in different ways. These are obtained empirically by graphing the ISO for each “vingtile” (the 19 slices of five percentiles from 5 to 95%). The value of $ISO(0)$, which can be erratic, is replaced by the average of $ISO(p=.45)$ and $ISO(p=.55)$. The shape of these curves can be explained by taxes, social and redistributive policies, and other empirical biases in the theoretical balance of power that can distort the income curve in such a way that the ISO is not constant. The poor can either benefit from income support or be the victims of extreme social exclusion. The rich can either organize a system of resource hoarding or accept the development of massive redistributive policies. Therefore, the hypothesis of the strict stability of α along the income scale generally does not hold, since power relations can be stronger or smoother at the top and bottom of the social ladder than near the median.

When the isograph is relatively flat (for example, Finland in 2004), α equals the Gini index (.24 for fi04 in Figure 1). In France, Germany and Brazil, the CF distribution hypothesis is an acceptable first-order approximation, but in other countries the isograph is obviously not constant. The isograph more often reveals a declining level of inequality at the top of the

distribution (an ISO with negative slope). An extreme case is Israel in 2010, with an ISO close to .50 at the middle of the distribution, similar to Brazil, but much lower at the ends. At the bottom 5% of the Israeli distribution, $ISO(-3) = .40$, which is similar to Spain and much less than the U.S. figure, and at the top 5% of the distribution the Israeli $ISO(3) = .36$, which is very similar to that in the U.S. These findings illustrate the large movements in local inequality over the income hierarchy. The crossings of the isographs for Israel and the U.S. show extreme inequality close to the median in Israel balanced by more equality at the extremes. The isograph helps us to compare inequalities that can shift over the income distribution.

4. The ABG method of the parametric estimation of the ISO

The shapes of the 232 isographs (Appendix 1: 232 Isographs) show that they can be accurately captured by only three parameters that I introduce here.¹³ The isograph shapes show that a coefficient pertaining to the level of local inequality close to the median (α) can be defined along with two shape parameters reflecting isograph curvature at the two extremes. Two correction coefficients β and γ are therefore determined, where $\alpha + \beta$ is the upper asymptote of the ISO and $\alpha + \gamma$ the lower asymptote. When γ and β are zero, the distribution is CF_α with coefficient $\alpha = \text{Gini}$. The added value of this method is to deliver unambiguous interpretable parameters of inequality showing both local inequality at the median, and corrections at the top and bottom of the income distribution.¹⁴

The parameterization proposed here is compatible with the well-established hypothesis that the upper tail has a power-tailed Pareto-type shape (Piketty, 2001), so that the upper asymptote of the $ISO(X)$ function should be a zero-slope line of the equation ($Y = \alpha + \beta$). We

¹³ Three plus one parameter of scale that disappears in the case of medianized incomes. GB2 and ABG have the same number of parameters, i.e., three of shape and one of scale.

¹⁴ This aspect is important: the GB2 distribution proposes, in general, a good fit of empirical distributions (Jenkins, 2009), but the interpretability of its p and q shape coefficients is unclear.

hypothesize, following Reed (2001), that the lower tail is also Pareto-shaped.¹⁵ Thus, the lower asymptote of ISO(X) should be the zero-slope line of the equation ($Y = \alpha + \gamma$). Between these two, we have smooth changes.

The parametric expression for these curvatures is based on two functions θ_1 and θ_2 related to hyperbolic tangent functions: $\theta_1(X) = \tanh(X/2)$ and $\theta_2(X) = \tanh^2(X/2)$ (see Figure 2).

PLACE FIGURE 2 HERE

We use two simple linear combinations of these θ functions, B and G, to make the coefficients easier to interpret. Consider the adjustment of ISO defined by:

$$\text{ISO}(X_i) = \alpha + \beta B(X_i) + \gamma G(X_i) \quad (4)$$

$$\text{where } B(X) = \frac{\theta_1(X) + \theta_2(X)}{2} \text{ and } G(X) = \frac{-\theta_1(X) + \theta_2(X)}{2}$$

$$\text{and } \theta_1(X) = \tanh(X/2) \text{ and } \theta_2(X) = \tanh^2(X/2)$$

$$\text{then, } M_i = \alpha X_i + \beta B(X_i)X_i + \gamma G(X_i)X_i \quad (5)$$

where $X_i = \text{logit}(p_i)$ and $M_i = \ln(m_i)$

Equation (5) is estimable as X_i and the functions are known, and there are no collinearity issues. The α , β , γ can be estimated in a single multivariate OLS regression without a constant. In the results:

- the coefficient α measures inequality close to the median;
- β characterizes the additional inequality at the top of the distribution, β being positive when the rich are richer than in the CF_α , so that the upper tail is stretched; and

¹⁵ This hypothesis will at some point have to be tested, along with the Milanovic *et al.* (2011) hypothesis that the vital subsistence minimum is at \$PPP 300 per year (in 1990 prices).

- γ characterizes the additional inequality at the bottom of the distribution, with γ being positive when the poor are poorer than in the CF_α .

The first comparative example refers to the estimation of the ABG (α β γ) coefficients on Israeli and U.S. data in 2010. In each sample, individuals are defined by their logit(quantile) of income and their related B and G functions. The OLS linear regression we propose is easy to carry out and produces the estimates of the ABG parameters and their standard errors in Table 2.¹⁶ The results reveal that at the middle of the distribution, there is more inequality in Israel than in the U.S., but the negative coefficients on the curvature parameters β and γ show that there is less inequality in Israel at the extremes than in the U.S. ($\alpha + \beta$ and $\alpha + \gamma$ are both smaller in Israel). These results reflect the complex comparison of the U.S. to Israel in Table 1 above, and underline the particular polarization in Israel (García-Fernández *et al.*, 2013); conversely, in the U.S., there is extreme inequality at the bottom (high values of gamma) with very low values for the poorest centiles.

PLACE TABLE 2 HERE

More generally, when β and γ equal zero, the distribution is a CF_α , where α = the Gini coefficient. In the empirical analysis of 232 cases, β and γ are always much smaller than α (Appendix 2: Table of 30 inequality indices): in this case, the CF distribution is an acceptable simplified first-order hypothesis, and β and γ are correction coefficients. When β (γ) is 1% higher, the ISO(X) function increases by 1% at the upper (lower) asymptote. The

¹⁶ To control for the potential problem of outliers, the regression interval is reduced to $abs(X) < 4$, which means the 2 centiles at both extremes are excluded from the regression. A second cut-off estimation has an alternative span of $abs(X) < 8$, which excludes a proportion of 5 out of 10.000 at both extremes. The results for α are not affected (the correlation between the two series is $r = 0.9998$), and those on β and γ are also stable ($r = 0.9886$ and $r = 0.9694$, respectively). This choice does not then particularly affect the results. Furthermore, in this case the r^2 are comparable, even though in the general case of regressions omitting the constant term, the ratio between the regression sum of squares and the total sum of squares makes no sense: if the constant term is omitted, the observed average and the fitted one differ. Here is an exception since, for the observed and for the estimated series, at the median level, both $logit(p_i)$ and $ln(m_i)$ are null: the log of the medianized median income is 0 and the logit-rank of the median is 0.

ABG has three shape parameters, plus one scale parameter derived from the ISO(X) estimation function (6).¹⁷

In this decomposition, α , $\alpha + \beta$ and $\alpha + \gamma$ are the inequality measures at the median, top and bottom of the distribution, respectively, and are homogeneous with the Gini coefficient in the sense that the upper tail of a distribution of coefficient ($\alpha + \beta$) is similar to a $CF_{\alpha+\beta}$.

There is no analytic expression for these measures since they come from a regression of M on the functions XB(X) and XG(X). Similarly, equation (5) yields no simple cumulative distribution function (cdf), but when $\beta = \gamma = 0$ equation (5) corresponds to a CF_{α} distribution.

The solutions are numerical whenever β or γ is non-zero. Here, the CF can be understood as a starting point with strong theoretical support (see above) that needs to be pragmatically adapted to the complex realities of tax and transfers, and social power imbalances that generate curvature at the top and the bottom: the empirical situations are (more or less) far removed from the microeconomic equilibrium (Parker, 1999).

The coefficients α , β and γ satisfy most of the criteria of the appropriate inequality measures (see Jenkins, 1991, 1995, Cowell and Jenkins, 1995, Haughton and Khandker, 2009: 105 sqq.):

- Mean independence: a proportional change in incomes does not affect the measures.
- Population-size independence: all else equal, a change in population size does not affect the measures.
- Symmetry: if individual (a) and (b) exchange their income levels, the measures are not affected.

¹⁷ In the conventional literature, this is a 4-parameter distribution, but with medianized income the traditional b coefficient is automatically set to 1.

- **Statistical testability:** it is possible to edit the confidence intervals of the OLS of (5) so that we can statistically test differences in estimated parameters that are useful for comparison purposes.
- **Decomposability:** this possibility is not exploited in the limits of the current paper, but covariates can be added to model (5) so that nested models can show how inequality results from inter- or intra-group variance, with the “group” being potentially defined by gender, education, ethno-cultural origins, and so on.
- **Pigou-Dalton Transfer (PDT) sensitivity:** the ABG method and the idea of local inequalities is not compatible with the strict PDT principle which claims that inequality falls when a richer individual (a) gives a part of her income to a poorer individual (b), provided that the hierarchy is not inverted. If (a) and (b) are above the median, and if the local inequality between (a) and (b) falls, inequality between the median and (b) increases since (b) gets richer, and thus further from the median. Such a transfer is ambiguous at the local level: even if the stretching between (a) and (b) is lower, meaning less inequality, the stretching between (b) and the median increases, meaning more inequality. The ABG method does satisfy, in any case, a weaker form of the PDT principle provided that (a) is above the median and (b) below it and that they remain in this order relative to the median after the transfer.

5. The comparative analysis of 232 datasets and inequality measures

Sections 5 and 6 analyse the performance of the ABG method compared to existing indicators (Section 5) and to the well-known GB2 distribution (Section 6). The added value of the ABG method over other measures is illustrated via its comparison to more customary inequality indices on a set of 232 harmonized microdata files covering 41 countries provided by the LIS datacenter project.¹⁸ This data source is very frequently used in the analysis of socioeconomic

¹⁸ This international consortium archives and harmonizes income-relevant datasets in the Western developed world and elsewhere, and is devoted to the microdata-based analysis of the inequality in disposable incomes after

inequality (Brandolini and Atkinson, 2001, Gornick and Jäntti, 2013), and the data set can be used as a large sensitivity test for the three indicators. It covers a large proportion of advanced countries plus some emerging countries (e.g., Brazil, China, India and Mexico).

The first result is that the absolute values of β and γ are small compared to that of α , so that $\alpha + \beta$ and $\alpha + \gamma$ are always in the interval $[0,1]$. The signs of β and γ can be positive or negative (Figure 3), and the point $\beta=0$ and $\gamma=0$ is in the middle of the range of β s and γ s (which confirms that the CF is like a base distribution, which tax and transfer policies, and the relations of power at different levels of the income distribution, can curve in different ways). A simple empirical typology based on the signs of β and γ is set out in Table 3.

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We can compare the three ABG indices to other standardized inequality measures (Jenkins, 1999/2010, Abdelkrim and Duclos, 2013). These selected indicators are well-known or based on simple income ratios. We consider added ISO indicators at five different levels. In addition, the size (as a proportion in the total population) of five income classes are included: the poor (po), lower middle class (mcl), middle class (mc), upper middle class (mcu), and the rich (ri). Overall, our analysis covers a set of 30 variables and 232 data samples (Appendix 2).

- **ABG class:**
 α, β, γ ; i.e., the three coefficients from the ABG method.

taxes and social transfers. The LIS income variable analyzed here is “dhi”: the total monetary current (yearly) income net of income taxes and social-security contributions. While some datasets are questionable either because of documented sources of bias which impair the possibilities of comparison or because the comparison shows that some cases are unexplainable outliers, the 232 samples available at the time of these empirical analyses (18/09/2014) are of particular interest due to the empirical diversity of cases they represent. The codes of the samples in the LIS data center correspond to the standard ISO 2-digit country codes followed by the 2 final digits of the year.

- **Atkinson class:**
a2, a1, ahalf = Atkinson class of indices, respectively with parameters 2, 1, and $\frac{1}{2}$ (Atkinson, 1970, also see Yitzhaki, 1983), the higher parameter (2) overweights the bottom of the distribution.
- **Generalized entropy class:**
ge2, ge1, ge0, gem1 = Generalized entropy class of indices, respectively with parameters 2, 1, 0, -1 (Berry et al., 1983). The lower parameter (-1) implies a focus on the bottom of the distribution.
- **Gini inequality index:**
The value of the standard Gini index (Gini, 1914).
- **Wolfson polarization index:**
The Wolfson index (Wolfson, 1986) of polarization.¹⁹
- **Foster-Greer-Thorbecke poverty class:**
fgt0, 1 and 2 show the Foster-Greer-Thorbecke (Foster et al., 1984) poverty index, with respectively parameters 0, 1, 2, and the poverty threshold of 60%. The higher the parameter, the greater the focus on very low incomes.
- **Income ratios:**
 - p90p50 = decile9/median: this measures inequality at the top.
 - p50p10 = median /decile1: this measures inequality at the bottom.
 - pp907550 = (decile9/quartile3) / (quartile3/median): this measures the degree to which inequality accelerates near the top decile, compared to the degree of inequality between the median and the top quartile. This corresponds to over-elongation at the top.
 - pp251050 = (quartile1/decile1) / (median/quartile1): this measures the degree to which inequality accelerates at the lower decile, compared to the degree of inequality at the lower quartile. This corresponds to over-elongation at the bottom.
- **ISO(X) class of measure of inequality:**
iso2, iso6, iso10, iso14, iso18 are respectively the values of ISO for the “vingtiles” (5% slices) 2, 6, 10, 14 and 18. These correspond to the values of X close to -3, -1, 0, +1 and +3, respectively.
- **Income class proportions:**
po, mcl, mc, mcl, mcu, ri. These are respectively, the proportion of the poor (medi < .5), lower middle class (.5 <= medi < .75), middle class (.75 <= medi < 1.25), upper middle class (1.25 <= medi < 2) and rich (2 <= medi) in the total population.²⁰
- **Income class based indicator of polarization:**
rpol = (mcl + mcu)/mc. This “polarization ratio” assesses the size of the lower and upper middle classes compared to the middle class, who are close to the median.

One important question is the relative position of the ABG parameters in the field of

inequality measures. A first answer is given by an analysis of the correlation matrix of these

¹⁹ The Wolfson index is chosen here because it is standard in the literature, even though more reliable propositions exist (Alderson et al., 2005, Chakravarty and D’Ambrosio, 2010).

²⁰ A log-symmetric definition such as .75 to 1.33 might be preferred, but the .75 to 1.25 of the median definition is far more common in the literature (Pressman, 2007). Working on quintile dynamics, Dallinger (2013) found similar variations in the different sub-strata of the middle classes.

indicators (Appendix3: the general correlation matrix of 30 inequality indicators): there is a very strong relation between α and the Gini index ($R = +.95$) thus confirming the relation of these two inequality measures when the CF_α approximation is acceptable. More generally, most of the measures correlate well with α . This is good news for the ABG method, but then what is its intrinsic added value? A second answer is that we also see interesting correlations for the β and γ coefficients, which thus provide complementary information to α : the degree to which inequality moves at the top and at the bottom of the distribution. A third more systematic answer comes from the principal component analysis (PCA) of the whole table (Figure 4 depicts the correlation circle). The PCA is a type of factor analysis²¹ used for quantitative measures, and its application to our indicator set (in Table 4) helps us to understand the multidimensional relations between these indicators. The first axis of the PCA (69% of the total variance) reveals the similar nature of many inequality measures, including α ; this axis picks up inequality intensity. The α coefficient appears on the first axis of the PCA, along with the Atkinson parameters 1 (a_1) and $\frac{1}{2}$ (a_{half}), the generalized entropy parameters 1 (ge_1) and 0 (ge_0), the Gini coefficient, a number of quantile ratios, as well as the Wolfson polarization index. This confirms that α is a new inequality parameter which is highly correlated with the main inequality measures, but is more sensitive (like the Wolfson index) to the median of the distribution.

PLACE TABLE 4 HERE

The role of β and γ becomes apparent on axes 2 and 3 (12% and 7% of the variance, respectively), which reveal the shape of inequality but not its intensity.

²¹ In the social sciences, PCA is a very common tool for the multidimensional descriptive synthesis of continuous variables (Everitt and Dunn, 2001: chap. 3, 48sq.). PCA extracts (via the diagonalization of the correlation matrix of “active variables”, here the selected inequality indicators) a hierarchy of complementary axes 1, 2, 3, etc. from higher to lower levels of variance. Figure 4 presents the scores (correlations) between axes 2 and 3 and the indicators.

- On the second PCA axis, β and γ are strongly correlated in the same direction as pp251050 and pp907550, the two measures of the over-elongation of the extreme deciles compared to the quartiles. The correlation with mcu and mcl (respectively, the upper and lower middle classes) is negative: the elongation at the top (resp. bottom) implies a smaller upper (resp. the lower) middle class that is stretched out. Positive values on the second axis reflect greater inequality at the extremes. Here, the generalized entropy index with parameter 2 is more strongly correlated on axis 2 than are the other traditional measures.
- Axis 3 shows the difference between γ and β , along with the contrast between pp251050 and pp907550. On this axis, the generalized entropy index with parameter 1 (gem1) and the Atkinson index with parameter 2 (a2) are located on the same side of axis3 as γ . All of these indicators are relatively more sensitive to inequality at the bottom. Conversely, the generalized entropy index with parameter 2 (ge2), located on the same side of axis3 as β , is sensitive to inequality at the top. Therefore, β and γ pick up salient features of the distribution that are less-well detected by other measures.

PLACE FIGURE 4 HERE

PLACE TABLE 5 HERE

The results here confirm that the estimated ABG parameters reflect central features of empirical distributions, and help us to understand the role played by other indicators. Table 5 uses the results from our 232 samples to shed light on the relation between the Gini index, the Atkinson 2 index, the generalized entropy 2 index and the ABG coefficients.

- The Gini index is very similar to α and is also correlated with the values of β (showing inequality at the top), but has almost no relation to γ . As a measure of inequality, the Gini index is (1) sensitive to the median (as is α), and (2) rich-oriented (like β).
- The Atkinson 2 index is more sensitive to lower-tail inequality. In the Atkinson 2 regressions γ and α are very significant, but β is not: the Atkinson 2 index is sensitive to both poverty and general inequality (the Gini coefficient).
- The generalized entropy 2 index is correlated with both β and α .

This analysis of correlations then suggests that the triple ABG parameters can be seen as contenders for the three coefficients of the Gini, Atkinson 2 and Generalized entropy 2 indices (GA2GE2). To see which triple performs best, we consider nested models of the five income-class proportions (po , mcl , mc , mcu , ri). Table 6 compares the goodness of fit (in terms of Δr^2) when ABG is first and GA2GE2 second, and vice versa. This comparison shows that the ABG triple always outperforms the GA2GE2 triple, with the advantage of ABG being particularly striking for the explanation of mcl and mcu , the lower and upper middle class respectively. In these 232 cases, ABG generally outperforms the GA2GE2 triple in terms of the prediction of income-class size.

We can also ask whether the ABG method provides a better assessment of polarization than the Wolfson index (Wolfson, 1986). The Wolfson index was developed from the Gini index, improving its sensitivity to median stretches when the other indices remain almost unchanged. Here, the ratio $r_{pol} = (mcl + mcu)/mc$, as defined earlier, should rise with polarization. The linear correlation matrix in Table 7 shows that, in terms of r^2 , the Wolfson index is indeed better than the Gini in predicting the r_{pol} ratio, but α is even better.

PLACE TABLE 6 HERE

PLACE TABLE 7 HERE

We now consider a nested model comparison of the 232 datasets with respect to middle-class polarization (r_{pol}). When entered first, the Gini coefficient explains more than half of the variation in r_{pol} , with the Wolfson adding a further 4.2%, which is significant; when the Wolfson index is entered first, the r^2 is 72.2%, with the Gini adding no further significant explanatory power. The Wolfson index does therefore act as a good measure of polarization, although to this extent α performs better (see Table 8). In general, for the different aspects of inequality measurement, the ABG method offers interpretable parameters that generally outperform the other methods in terms of the description of the distribution and the size of income classes.

PLACE TABLE 8 HERE

6. How do the ABG-distribution and GB2 perform?

Another aspect of the ABG method is its distributional shape: the three parameters describe a distribution that is tailored to fit the observed data. How does ABG perform in this respect? In the contemporary income-distribution literature, the GB2 is the leading contender for the best measure (Jenkins, 2009, Graf and Nedyalkova, 2014). This distribution is particular in the universe of Beta-type laws: it is the most general, as many other distributions are special cases. It has 4 parameters, one of scale (b) and three of shape (a, p, q), which is the same number as the ABG distribution, provided that we consider equation (5) above as a general expression of an empirical distribution where the fourth parameter (size) is the median. In terms of microeconomic theory, the GB2 results from a simple model of firm behavior

(Parker 1999), and is acknowledged for its flexibility. Statistical tools to estimate the GB2 parameters are easily available.²²

To compare the respective performances of ABG and of GB2, we consider the divergence from the empirical observed distribution (OBS) of ABG, GB2 and CF. This is not an easy task since the GB2 predicted values are based on a known cumulative distribution function (cdf) and an unknown quantile function (although its estimation via simulation is possible) and the ABG provides a quantile function rather than a cdf. Our solution here is to compare the predicted values of each “vingtile” level of logged incomes for four quantile functions: the empirical distribution as the target, the GB2 and ABG as competitors, and the CF_α with $\alpha =$ Gini as the baseline. As they have more parameters (and are thus more flexible), the GB2 and ABG provide a better fit to the OBS than does CF_α . One measurement of the goodness of fit is the r_a^2 (the adjusted coefficient of determination): the higher is r_a^2 , the better the fit to the OBS. The most difficult issue concerns the estimation of the GB2 parameters (a, b, p, q) for the 232 samples; here the STATA `gb2lfit` program only converged quickly in 205 cases. The maximum number of iterations was set to 6, since convergence after 7 or more iterations are exceptional and may be considered as outliers.

PLACE TABLE 9 HERE

Our analysis is restricted to the 205 convergent cases. We have for each country the vectors of 19 vingtiles of log income levels for the CF-Gini ($linc_q$), ABG ($lina_q$), GB2 ($ling_q$), and the empirical OBS distribution ($lino_q$: o for observed), with $q = 1 \dots 19$. The adjusted r_a^2 of the OLS regression of ($linc_q$) on ($lino_q$) reflects the quality of the CF hypothesis: the higher the value, the better the fit (Table 9). On average, the CF is a good first-order approximation ($r_a^2 =$

²² In particular, Jenkins' (2014) STATA based component (`ssc install gb2lfit`) provides an estimate of the a, b, p, q parameters, as well as additional information such as the predicted quantiles. The previous `gb2fit` program exhibited more convergence problems.

.996), and both GB2 and ABG improve the fit further, with a clear advantage for the latter. In 67.8% of cases, GB2 is better than CF, but AGB outperforms CF in 85.8% of cases and GB2 in 76.5%.

We can explain the better fit of the ABG methodology. We simulated many GB2 distributions from the shape parameters a , p and q , each randomly-defined; we then fitted these with the ABG method, and found no cases where the β and γ coefficients were strongly negative at the same time. This means that strongly polarized distributions such as than in Israel in 2010, with its stretched middle class and relatively more equality at the top and the bottom, cannot be generated from the GB2 distribution. The GB2 is flexible, but does not cover every case, and in particular those of type 2 of the typology in Table 2 above. This means that the GB2 with parameters a , p , q is less general than the ABG with coefficients α , β and γ , which is more flexible with the same number of parameters.

The GB2 is a good tool, and has the advantage of being theoretically more solid and mathematically purer than the ABG, but does nonetheless present some difficulties. The interpretation of the GB2 parameters a , p , q is not obvious, with the exception of the case where $p=q=1$. The ABG method is on the other hand less theoretically-satisfying: it has no simple analytical expression, is very empirical, and is a computer-oriented fitting tool.

However, ABG produces three easy to estimate and interpret coefficients that make sense of the distribution of inequality, with values that are compatible with the Gini tradition since $\alpha = \text{Gini}$ if β and γ are close to 0.

7. Representing the shape of the income distribution: the strobiloid

The ABG decomposition provides a method for smoothing the empirical quantile distribution function. If, for instance, we are interested in the architecture of societies represented by the distribution density curve, as in the seminal work of Pareto (1897: 315), we can plot income

on the vertical hierarchical axis and the density value on the horizontal axis, as in Figure 5.

One convenient way of standardizing the representations, for comparison purposes, is to normalize the income curve. With both the medianization of income and the normalization of the surface to 1 (so that it defines the density of the distribution), we can superpose the curves for different periods or countries. This is the strobiloid representation (Chauvel, 1995, Lipietz, 1996, Chauvel, 2013).²³

PLACE FIGURE 5 HERE

These empirical strobiloids reveal the diversity of income distributions across countries and reflect the change in socioeconomic architecture within countries. In the strobiloid, the wider the curve, the more individuals there are at this level of the graph: middle-class societies will have a large belly (Denmark), whereas in the contemporary American distribution a large proportion of the population is close to the bottom. Kernel smoothing can produce similar curves, but the ABG method relies on a Pareto power-tail compatible methodology to produce interpretable parameters.²⁴ This new tool allows the country and time comparison of the considerable developments in the intensity and shape of inequality.

PLACE FIGURE 6 HERE

The strobiloid shows that incomes in Denmark in 1987 are generally “more equal” than elsewhere, although the particularity of Denmark (due to its low α and high positive γ) is its lack of rich rather than its lack of poor, with some of the latter being stretched far to the

²³ The strobiloid is based on Pareto’s idea (1897:313) that the shape is one of an arrow or of a spinning top. This representation, which is similar to Pareto’s first representations of the income pyramid, allows us to make 2-by-2 comparisons over countries, time, etc. Nielsen (2007) provides an overview of Pareto’s legacy, and considers why this has generally been neglected in the social sciences.

²⁴ Kernel density analysis is generally unable to provide a correct assessment of the extremities of the curves.

bottom of the distribution. The bottom part of the curve in Germany in 1983 shows the same level of inequality as in Denmark in 1987 (as can be seen from the isograph in Figure 6), although there is more inequality in Germany in 1983 for higher income levels. In terms of public policy, the structure of Germany in 1983 is a particular model of homogeneity below the median with a high implicit level of minimum income.

The French distribution is fairly common in Europe and is stable over the period under consideration. On the contrary, there is a strong polarization trend in the U.K., which is converging to the onion-shaped strobiloid of the U.S. The U.S. itself has an even more pronounced onion shape with increasing inequality. One feature of this shape is less the extreme values at the top but rather the lower values with a very high γ . Israel, the final case, may be the most symbolic in terms of the shift from a more equal to a far less equal distribution, with one particular feature: a steady decline in the median class of incomes with a relatively strong minimum-income scheme, leading to the development of an unprecedented arrowhead-shaped curve. Israel appears then as an extreme case of rapid polarization over recent decades, which is confirmed by the isograph in Figure 6. A broader international comparison reveals the diversity of distributions across countries (Appendix 4: strobiloids in 32 countries).

8. Conclusion: The added value and further extensions of the ABG method

The ABG methodology represents progress in terms of both measurement and graphical representation (CF curve, isograph and strobiloid) of the diversity of inequality at different levels of income, since in many cases inequality is anisotropic along the income scale. In terms of public policies, it can reveal useful information about the different dynamics of

inequality, where inequality at the median, α , can be analyzed in parallel with that at the extremes described by β and γ .

The ABG approach relies on an easy-to-use family of distributions to model income distributions. It can be used, for example, to model extremely unequal distributions such as Zipf laws (Gabaix, 1999), which are extreme Pareto distributions with α close to 1. It also helps us to understand why the Gini coefficient can pose problems when the isograph is far from being a constant (when β and γ differ greatly from 0).

In this approach, the magnitudes of ranks and incomes, defined by $\text{logit}(\text{quantile})$ and $\text{log}(\text{income})$, are almost linearly-related. The $\text{logit}(\text{quantile})$ may therefore be an important tool for the measurement of inequality, and could be used in other fields such as income mobility. The further development of the ABG should include the analysis of statistical significance and group decomposability. As the ABG coefficients come from linear regressions, we can add control variables to understand how the gaps between groups (education, gender, etc.) contribute to overall inequality. Last, we also need further analysis of isograph shapes when the absolute value of X is over 5, for the very rich and very poor.

The results that we presented here can also be found with more traditional tools, but the α , β , γ ABG method, the CF and the isograph, and the associated strobiloid, represent more systematic and easier to use tools for the detection of particular shapes, propose better measures of the income distribution, and help us to better understand the anisotropy of inequality.

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Table 1: Percentiles of Incomes in Israel and the U.S. in 2010 and the Difference between Them

	p1	p5	p10	p25	p50	p75	p90	p95	p99
il10	0.173	0.301	0.368	0.568	1.000	1.637	2.366	2.945	4.444
us10	0.057	0.235	0.362	0.611	1.000	1.531	2.171	2.731	4.501
Diff.	-0.116	-0.066	-0.006	0.043	0	-0.106	-0.195	-0.214	0.057

Note: Diff. shows the simple percentile level difference between the U.S. and Israel.

Table 2: Estimates of ABG Parameters in Israel and the U.S. in 2010

IL2010	Coefficient	S.E.	95% C.I. min	95% C.I. max
α	0.53852	0.00059	0.53737	0.53968
β	-0.23972	0.00124	-0.24215	-0.23728
γ	-0.14505	0.00114	-0.14728	-0.14282
N =	18,936	$r^2 =$	0.9959	
US2010	Coefficient	S.E.	95% C.I. min	95% C.I. max
α	0.42699	0.00005	0.42689	0.42709
β	-0.09251	0.00015	-0.09280	-0.09223
γ	0.05202	0.00031	0.05141	0.05263
N =	191,055	$r^2 =$	0.9991	

Table 3: Typology of Income Shapes

	γ negative	γ positive
β positive	Type 1: Rich are richer and the poor richer than under the CF. The isograph has a positive slope. 13 cases. Typical country: za08	Type 2: Rich are richer and the poor poorer, but the middle class is relatively homogeneous. The isograph has a U shape. 35 cases. Typical country: de04
β negative	Type 3: Rich are poorer and the poor are richer than under the CF. The isograph has an inverted-U shape. 83 cases. Typical country: il10	Type 4: Rich are poorer and the poor are poorer. The isograph has a negative slope. 101 cases. Typical country: us10

Table 4: PCA Scores: Correlation between the Principal Components and 30 Indicators of Inequality

Indicator	v1	v2	v3
α	0.2143	-0.1114	0.0138
β	-0.0548	0.4194	-0.244
γ	-0.0565	0.3174	0.4321
a2	0.1166	0.0958	0.3377
a1	0.2171	0.0666	-0.0154
Ahalf	0.2145	0.0875	-0.0738
ge2	0.1373	0.1738	-0.1968
ge1	0.2092	0.1123	-0.1184
ge0	0.2154	0.0879	-0.0416
gem1	-0.0002	0.131	0.2562
Gini	0.2174	0.0239	-0.0381
Wolfson	0.2183	-0.0179	-0.0404
fgt0	0.2183	0.0094	0.0044
fgt1	0.215	0.0889	0.0055
fgt2	0.2096	0.123	0.0089
p90p50	0.2108	0.0632	-0.1424
p50p10	0.2098	0.0245	0.1522
pp907550	-0.0266	0.3587	-0.355
pp251050	-0.0451	0.3495	0.3557
iso2	0.2096	-0.0046	0.1789
iso6	0.2118	-0.103	0.0745
iso10	0.2115	-0.1105	0.0226
iso14	0.2148	-0.0662	-0.0305
iso18	0.2147	0.0142	-0.1062
Po	0.2085	-0.0219	0.1833
Mc	-0.2015	0.1644	-0.0753
Mcl	-0.0646	-0.3677	-0.2298
Mcu	-0.1162	-0.298	0.2478
Ric	0.2146	-0.0182	-0.0723
Rpol	0.1859	-0.2517	0.0745

Table 5: OLS Coefficients: Regression of the Gini, Atkinson 2 and Generalized Entropy 2 indices on the ABG Coefficients

	Coefficient	S.E.	T	P > t	95% C.I. min	95% C.I. max
Gini index $r^2 = 0.9849$						
α	0.8978	0.0076	117.9	0.0000	0.8828	0.9128
β	0.4839	0.0160	30.2	0.0000	0.4523	0.5154
γ	0.1192	0.0141	8.4	0.0000	0.0913	0.1471
Cons.	0.0277	0.0024	11.3	0.0000	0.0228	0.0325
Atkinson 2 $r^2 = 0.3964$						
α	1.4425	0.1322	10.9	0.0000	1.1820	1.7030
β	0.0293	0.2778	0.1	0.9160	-0.5182	0.5767
γ	1.9340	0.2457	7.9	0.0000	1.4500	2.4181
Cons.	-0.0506	0.0425	-1.2	0.2350	-0.1344	0.0331
Generalized entropy 2 $r^2 = 0.4131$						
α	3.0979	0.2582	12.0	0.0000	2.5891	3.6067
β	3.8710	0.5426	7.1	0.0000	2.8018	4.9403
γ	-0.0388	0.4798	-0.1	0.9360	-0.9842	0.9067
Cons.	-0.5315	0.0830	-6.4	0.0000	-0.6951	-0.3680

Note: VIF < 1.28; N = 232

Table 6: R² Added Value in Nested Models of Income-Class Proportions of the ABG Coefficients and the GA2GE2 Triple Coefficients (Gini Index, Atkinson 2, Generalized Entropy 2)

	ABG first	GA2GE2 delta r^2	Improvement t sig. p		GA2GE2 first	ABG delta r^2	Improvement t sig. p
Po	0.9775	0.0011	0.0110		0.8855	0.0931	0.0000
Mcl	0.3642	0.0460	0.0007		0.1139	0.2963	0.0000
Mc	0.9231	0.0091	0.0000		0.8668	0.0654	0.0000
Mcu	0.4566	0.0490	0.0001		0.3753	0.1302	0.0000
Ri	0.9837	0.0006	0.0400		0.9655	0.0187	0.0000

Table 7: Correlation between the Ratio of Polarization, Gini, Wolfson Index and ABG coefficients

Var	Rpol	Gini	Wolfson	α	β	γ
Rpol	1					
Gini	0.8263	1				
Wolfson	0.8499	0.9831	1			
α	0.9197	0.9524	0.9778	1		
β	-0.5652	-0.1603	-0.2373	-0.4252	1	
γ	-0.3861	-0.2392	-0.3229	-0.3784	0.3605	1

Table 8: R² Added Value in Nested Models of Rpol (Middle Class Polarization) of the Gini Coefficient and Wolfson Index

	Gini first	Wolfson delta r ²	Improvement sig. p		Wolfson first	Gini delta r ²	Improvement sig. p
Rpol	0.6828	0.0422	0.0000		0.7224	0.0026	0.1430
	α first	Wolfson delta r ²	Improvement sig. p		Wolfson first	α delta r ²	Improvement sig. p
Rpol	0.8458	0.0556	0.0000		0.7224	0.1789	0.0000

Figure 1: The Isograph in 10 contrasting cases

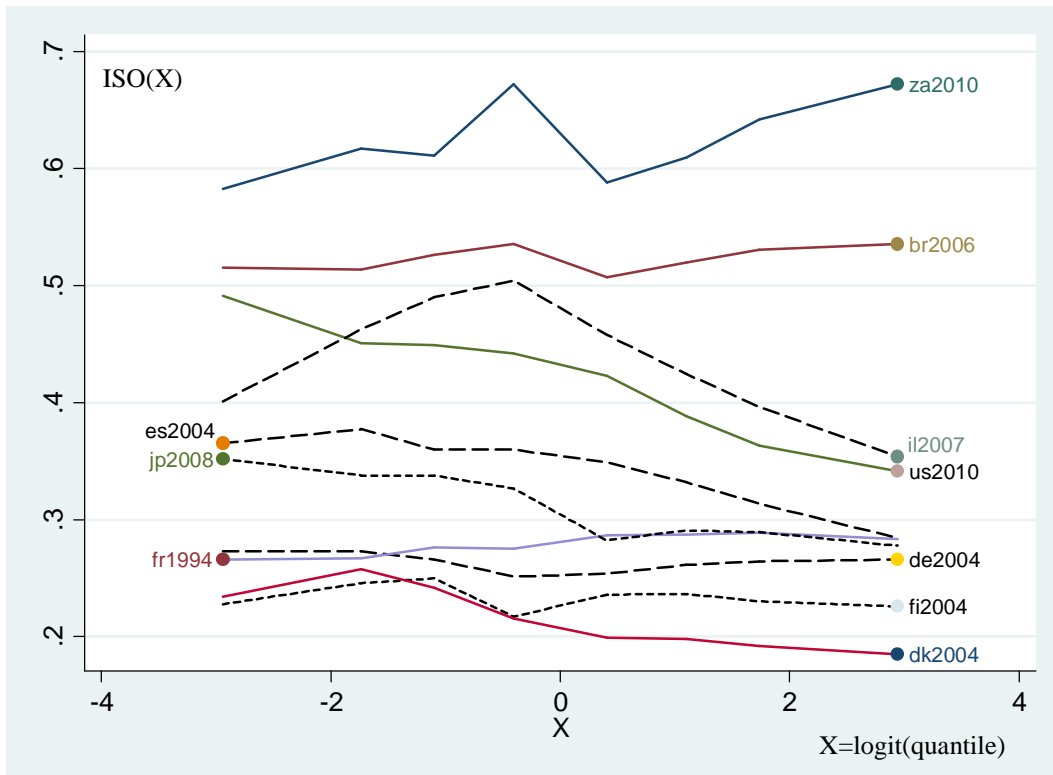


Figure 2: The θ_1 and θ_2 functions

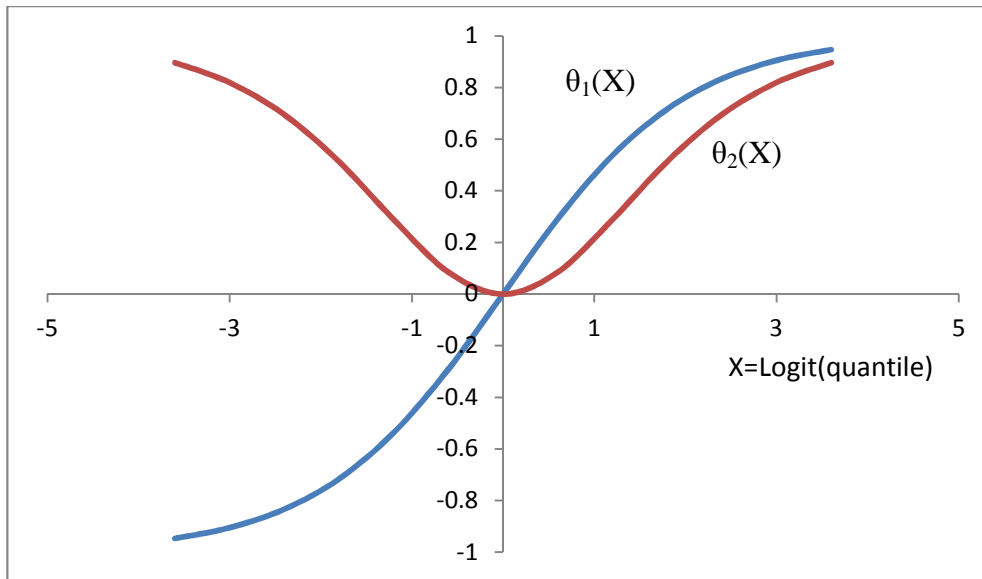


Figure 3: The relation between β and γ

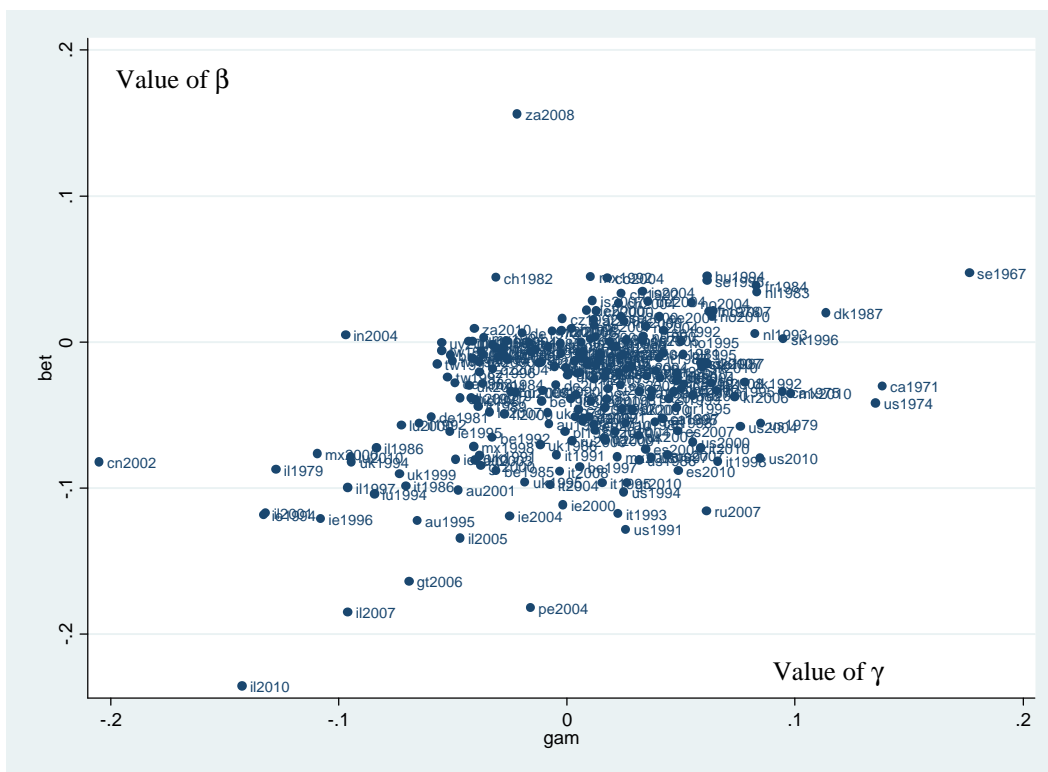


Figure 4: The unrotated PCA components of the 30 indicators of inequality (PCA scores)

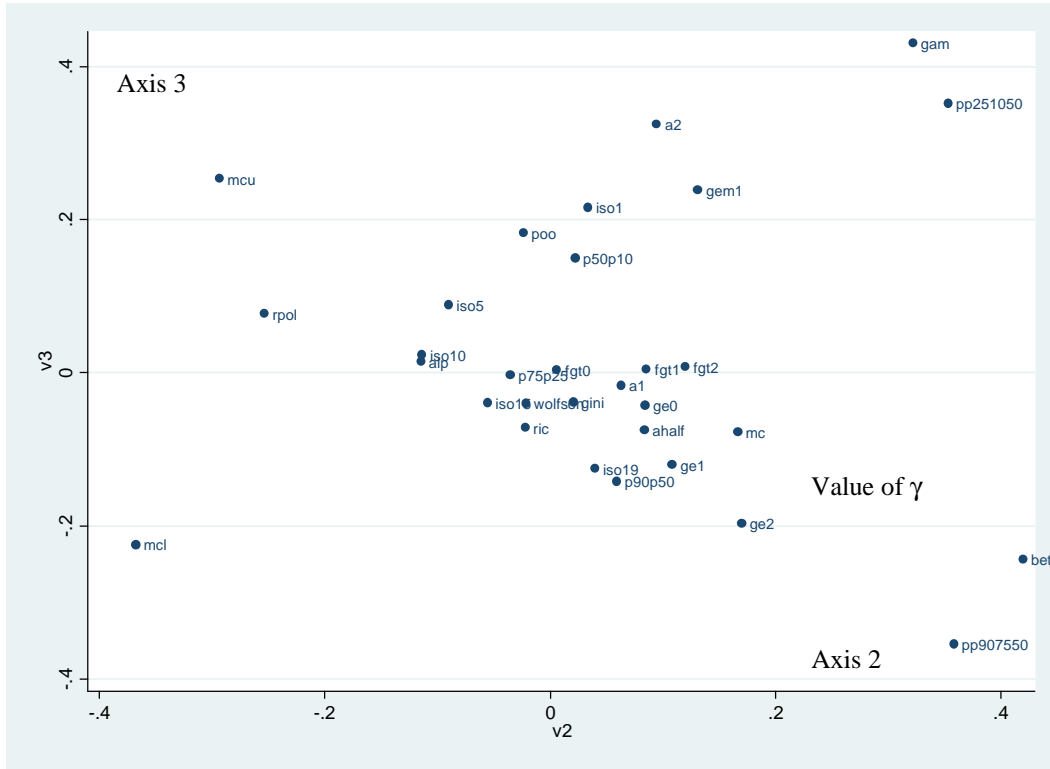
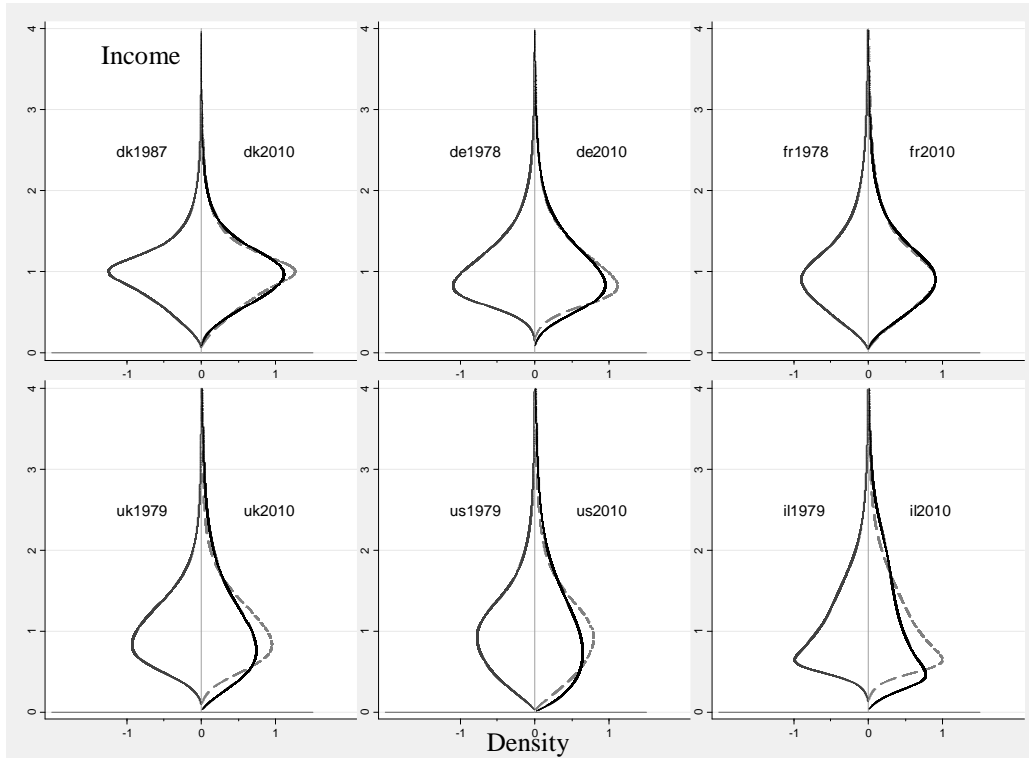


Table 9: The frequency of a better fit of distribution D1 compared to D2 (%) on 205 Samples (27 excluded cases with more than 6 iterations)

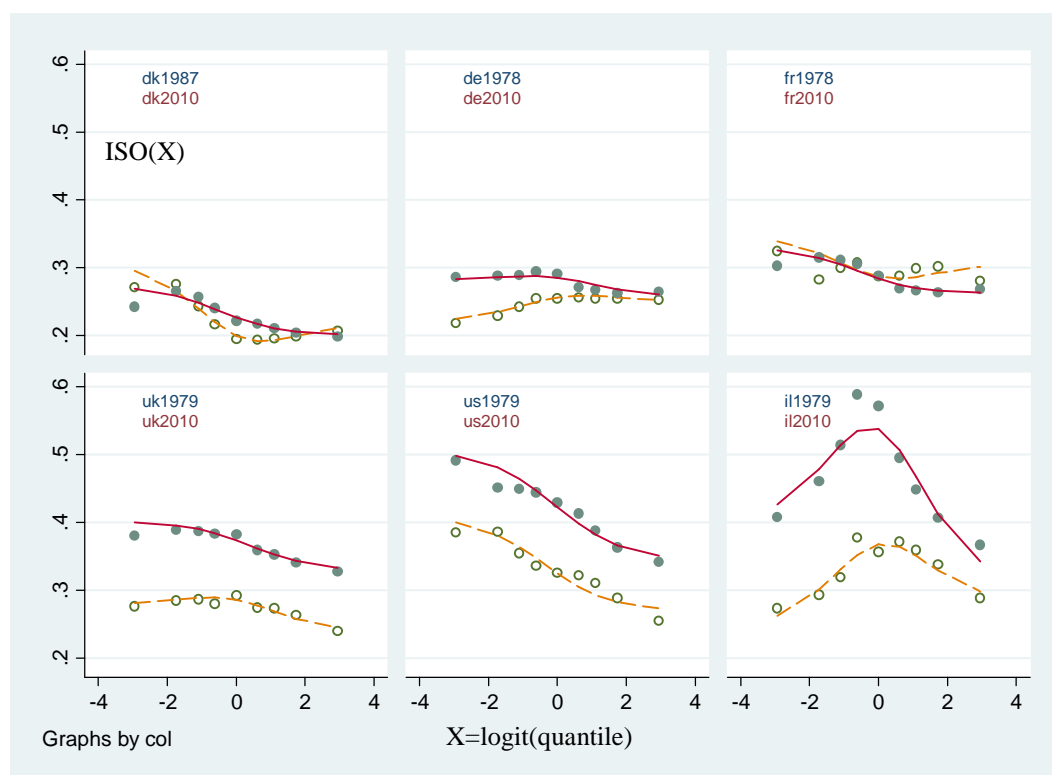
	Average adjusted r^2 of the fit of OBS	Pair comparison: % of cases where the fit of A is worse than that of B
CF	0.9959	CF worse than GB2: 67.8%
GB2	0.9975	GB2 worse than ABG: 76.5%
ABG	0.9989	CF worse than ABG: 85.8%

Figure 5: Six typical strobiloids (Denmark, Germany, France, U.K., U.S. and Israel)



Note: The strobiloid shows the income hierarchy (on the vertical axis, 1 = median). The curve is larger (horizontal axis) when the density at this level of income is higher: Many individuals are at the intermediate level near to the median and their number diminishes at the top and at the bottom. Thus, in strobiloids with a larger belly, the intermediate middle class is larger with a more equal distributions.

Figure 6: The Isographs for six typical countries



Note: The dots represent the empirical values and the lines are the fitted isographs (ABG method). For each country, two periods are considered: the dashed line and white dots pertain to the older years, the full line and gray dots refer to more recent years. The higher the curve at a given level of X (logit rank), the greater are the income inequalities at this level. Israel over 1986-2010 is an obvious case of extreme polarization.

Supporting Information

Additional Supporting Information may be found in the online version of this article at the publisher's web-site:

Appendix 1: Figure of 232 Isographs (MS Word .doc)

Appendix 2: Table of 30 Inequality Indices (Stata 12 .dta)

Appendix 3: General Correlation Matrix of 30 Inequality Indicators (MS Excel .xls)

Appendix 4: Figure of 32 Countries Strobiloids (MS Word .doc)

Appendix 5: Distribution Simulator for ABG (MS Excel .xls)

These appendixes can be downloaded at <http://www.louischauvel.org/roiw.zip>

The abg method can be implemented in STATA with the command:

```
ssc install abg
```