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Skewness and Kurtosis Properties of Income Distribution Models

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ABSTRACT

This paper explores the ability of some popular income distributions to model observed skewness and kurtosis. We present the generalized beta type 1 (GB1) and type 2 (GB2) distributions' skewness-kurtosis spaces and clarify and expand on previously known results on other distributions' skewness-kurtosis spaces. Data from the Luxembourg Income Study are used to estimate sample moments and explore the ability of the generalized gamma, Dagum, Singh-Maddala, beta of the first kind, beta of the second kind, GB1, and GB2 distributions to accommodate the skewness and kurtosis values. The GB2 has the flexibility to accurately describe the observed skewness and kurtosis.

JEL: C16, C52, E25

Keywords: skewness, kurtosis, generalized beta type 2 distribution, generalized gamma distribution

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1. INTRODUCTION

Pareto's pioneering work in modeling the distribution of income was published more than a century ago. He observed that, in many cases, an approximately linear relationship existed between different income levels and the number of individuals receiving at least that level of income. While the Pareto distribution often provided an accurate model of the upper tail of the distribution, it did a poor job of describing the lower tail. Since inaccurate estimates of distributions can result in misleading policy implications, this led to the consideration of different distributions that more accurately modeled income. Gibrat's (1931) law of proportionate effect provided a theoretical foundation for the use of a two-parameter lognormal distribution, which was studied in more detail by Aitchison and Brown (1969). Battistin, Blundell, and Lewbel (2009) used the lognormal to compare the distribution of income and consumption across households. Other two-parameter models include the gamma (Salem and Mount, 1974) and the Weibull (Bartels and Van Metele, 1975). While these two-parameter models provide increased flexibility relative to single-parameter models, they do not allow for intersecting Lorenz curves, which frequently arise with income data.

Intersecting Lorenz curves can be obtained by adding a third parameter. Some common three-parameter models that have been used to model income and allow for intersecting Lorenz curves include the beta of the first kind (B1), the beta of the second kind (B2), the Dagum (DAGUM), and the Singh-Maddala (SM) distributions. Thurow (1970) used the B1 to explore explanatory factors associated with the distribution of income for whites and blacks in the United States. Chotikapanich, et al. (2010) used the B2 to analyze global income inequality. Dagum's (1977) distribution was based on theoretical foundations and provided a significant improved fit in many applications. Singh and Maddala's (1976) distribution also provided an improved fit relative to the two-parameter models previously considered. The generalized gamma (GG) is another three-parameter model that permits intersecting Lorenz curves and yields improved fit relative to the lognormal and gamma distributions.

The generalized beta of the first and second kind (GB1 and GB2, respectively) are four-parameter models that include each of the previously described models as special or limiting cases. McDonald (1984) provided an early reference to the GB1 and GB2 and its special cases, along with applications. Distributional characteristics, such as moments and the Gini, Pietra, and Theil measures of inequality, can be expressed in terms of the distributional parameters. Other distributions, such as the double-Pareto-lognormal distributional distribution which have desirable properties and provide an excellent fit to empirical fit to empirical data (Kleiber and Kotz (2003), Reed and Jorgenson (2004), and Reed and Wu (2008)), have been recently explored, the focus of this paper will be restricted to The GB1 and GB2 and its special cases.

There is a substantial literature describing the properties, estimation procedures, and applications of these distributions. Kleiber and Kotz's book (2003) provides an excellent summary of these issues and includes more than 500 references to the theoretical foundations and diverse applications of these and other distributions in economics and actuarial science.

Maximum likelihood estimation is a common method of estimating the parameters of income distributions, although other methods have been used. Income data is often reported in a grouped format. Estimation with grouped data can be performed by maximizing a multinomial likelihood function or minimizing a Chi-square goodness of fit statistic. Other estimators may be obtained by imposing restrictive assumptions (such as assuming that the observations appear at the midpoint of an income group) or by top coding—both of which ignore intra-group variability. These restrictions can impact estimator precision. Gastwirth (1972) studied the impact of grouping on estimating the Gini coefficient by deriving upper and lower bounds on the Gini coefficient. The lower bound assumes all incomes in an interval equal the average income, and the upper bound corresponds to distributing the income to maximize the spread within each group. McDonald and Ransom (1981) demonstrated that a failure to take account of sampling variation can lead to misleading results.

More recently, continuous income data have become increasingly available and have expanded possible estimation methods and analysis. These data include information drawn from the US Census

Bureau, Current Population Survey, and other sources, and they are readily available on the Internet. The use of continuous observations yields more accurate estimation of such descriptive statistics as skewness, kurtosis, and Gini coefficients.

In this paper we explore the ability of the GB1 and GB2 distributions to model skewness and kurtosis. While many of the theoretical results are available in different sources, we summarize, clarify and expand on previously known results, and derive new skewness-kurtosis spaces for the GB1 and GB2 distributions. In addition, we present previously unknown relationships between the skewness-kurtosis spaces for different distributions. We apply the results to the Luxembourg Income Study (LIS) for thirteen countries, three definitions of income, and two time periods. The GB2 provides the flexibility to model the observed skewness and kurtosis levels in the cases considered.

The next section summarizes basic characteristics of a number of popular distributions of income (for models of positive income only). Their respective skewness-kurtosis spaces are described in the Appendix. Some new results, corrections to previously published results, and known results are given. Section 3 reports the observed skewness and kurtosis values for different countries, definitions of income, and time periods and compares them to the permissible values based on the distributions considered. Section 4 summarizes our findings.

2. THE MODELS

Since many of the most commonly used models for the distribution of income are special cases of the generalized beta type 1 (GB1) and type 2 (GB2) distributions, we begin by defining them, their moments, and some of their special cases.

The GB1 and GB2 probability density functions (*pdf*s) are defined by

$$GB1(y;a,b,p,q) = \frac{|a|\, y^{ap-1}\left(1-\left(y/b\right)^{a}\right)^{q-1}}{b^{ap}B\left(p,q\right)}, \quad 0 < y < b$$

$$GB2(y;a,b,p,q) = \frac{|a|\, y^{ap-1}}{b^{ap} B(p,q)\left(1+(y/b)^{a}\right)^{p+q}}, \;\; 0 < y$$

with corresponding moments given by

$$E_{GB1}\left(Y^{h}\right) = \frac{b^{h}\Gamma(p+q)\Gamma(p+h/a)}{\Gamma(p+q+h/a)\Gamma(p)}$$

$$E_{GB2}\left(Y^{h}\right) = \frac{b^{h}\Gamma(p+h/a)\Gamma(q-h/a)}{\Gamma(p)\Gamma(q)}$$

The Pareto distribution can be viewed as a special case of the GB1:

$$\begin{aligned} Pareto(y;b,p) &= GB1(y;a=-1,b,p,q=1) \\ &= \frac{py^{-p-1}}{b^{-p}}, \;\; b < y \end{aligned}$$

as can the beta of the first kind (B1), used by Thurow (1970):

$$\begin{aligned} B1(y;b,p,q) &= GB1(y;a=1,b,p,q) \\ &= \frac{\mathrm{y}^{p-1}(b-y)^{q-1}}{b^{p}B(p,q)}, \;\; 0 < y < b \end{aligned}$$

The moments of the Pareto and B1 distributions can easily be obtained from expressions for the GB1 moments with appropriate substitutions.

The Singh-Maddala and Dagum distributions are obtained from the GB2 by substituting p=1 and q=1, respectively, into the GB2 *pdf* to obtain

$$\begin{aligned} SM(y;a,b,q) &= GB2(y;a,b,p=1,q) \\ &= \frac{aqy^{a-1}}{b^{a}\left(1+(y/b)^{a}\right)^{q+1}}, \;\; 0 < y \end{aligned}$$

$$\begin{aligned} DAGUM(y;a,b,p) &= GB2(y;a,b,p,q=1) \\ &= \frac{apy^{ap-1}}{b^{ap}\left(1+(y/b)^{a}\right)^{p+1}}, \;\; 0 < y \end{aligned}$$

The Dagum and Singh-Maddala distributions, respectively, are known as the Burr Type 3 and Burr Type 12 distributions in the statistics literature (Kleiber and Kotz, 2003).

The beta of the second kind (B2), used by Chotikapanich, et al. (2010), is another three-parameter special case of the GB2:

$$\begin{aligned} B2(y;b,p,q) &= GB2(y;a=1,b,p,q) \\ &= \frac{y^{p\text{-}1}}{b^p B(p,q)\left(1+(y/b)\right)^{p+q}},\ \ 0<y \end{aligned}$$

The moments of the SM, Dagum, and B2 distributions can easily be obtained from expressions for the GB2 moments with appropriate substitutions.

The generalized gamma (GG) was used by Kloek and Van Dijk (1978); Taillie (1981); McDonald (1984); Atoda, Suruga, and Tachibanaki (1988); and Bordley, et al. (1996) to study the income distribution in a number of different countries. The GG *pdf* is obtained from the GB2 by taking the following limit

$$\begin{aligned} GG(y;a,\beta,p) &= \lim_{q\to\infty} GB2\left(y;a,b=q^{1/a}\beta,p,q\right) \\ &= \frac{|a|\,y^{ap-1}e^{-(y/\beta)^a}}{\beta^{ap}\Gamma(p)} \end{aligned}$$

The moments of the GG can be expressed as

$$E_{GG}\left(Y^h\right) = \frac{\beta^h\Gamma(p+h/a)}{\Gamma(p)}$$

The gamma (GAM), Weibull (W), lognormal (LN), and power function (PF) *pdf*s are the following special or limiting cases of the generalized gamma:

$$\begin{aligned} GAM(y;\beta,p) &= GG(y;a=1,\beta,p) \\ &= \frac{y^{p-1}e^{-(y/\beta)}}{\beta^p\Gamma(p)} \end{aligned}$$

$$W\left(y;a,\beta\right)=GG\left(y;a,\beta,\,p=1\right)$$
$$=\frac{ay^{a-1}e^{-(y/\beta)^{a}}}{\beta^{a}}$$

$$LN\left(y;\mu,\sigma^{2}\right)=\lim_{a\to0}\mathrm{GG}\left(y;a,\beta=\left(a\sigma\right)^{2/a},\,p=\frac{a\mu+1}{a^{2}\sigma^{2}}\right)$$
$$=\frac{\mathrm{e}^{-\frac{\left(\ln(\mathrm{y})-\mu\right)^{2}}{2\sigma^{2}}}}{\sqrt{2\pi}\,y\sigma}$$

$$PF\left(y;\beta,\theta\right)=\lim_{a\to\infty}GG\left(y;\beta,\,p=\theta/a\right)$$
$$=\frac{\theta\mathrm{y}^{\theta\text{-}1}}{\beta^{\theta}},\ 0<\mathrm{y}<\beta$$

The moments of the gamma and Weibull distributions can easily be obtained from expressions for the GG moments with appropriate substitutions.

The moments for the LN and PF are given by

$$E_{LN}\left(Y^{h}\right)=e^{h\mu+h^{2}\sigma^{2}/2}$$

$$E_{PF}\left(Y^{h}\right)=\beta^{h}\left(\frac{\theta}{\theta+h}\right)$$

Equations for the Pietra, Theil, and Gini measures of inequality expressed in terms of the distributional parameters have been derived by various authors and are summarized in Kleiber and Kotz (2003) and McDonald and Ransom (2008) for the LN, GG, GB1, GB2, and special cases.

The purpose of this paper is to consider the ability of these distributions to model observed combinations of skewness and kurtosis arising in different income studies. We use the standardized skewness and kurtosis measures respectively defined by

$$\gamma_1 = \frac{E(Y-\mu)^3}{\sigma^3} = \frac{E(Y^3) - 3E(Y^2)\mu + 2\mu^3}{\sigma^3}$$
$$\gamma_2 = \frac{E(Y-\mu)^4}{\sigma^4} = \frac{E(Y^4) - 4E(Y^3)\mu + 6E(Y^2)\mu^2 - 3\mu^4}{\sigma^4}$$

where μ and σ^2 in these equations denote the mean and variance of the random variable of interest. Standardized skewness and kurtosis are often denoted by $(\sqrt{\beta_1}, \beta_2)$ in the literature, but the notation (γ_1, γ_2) more clearly allows for positive and negative skewness.

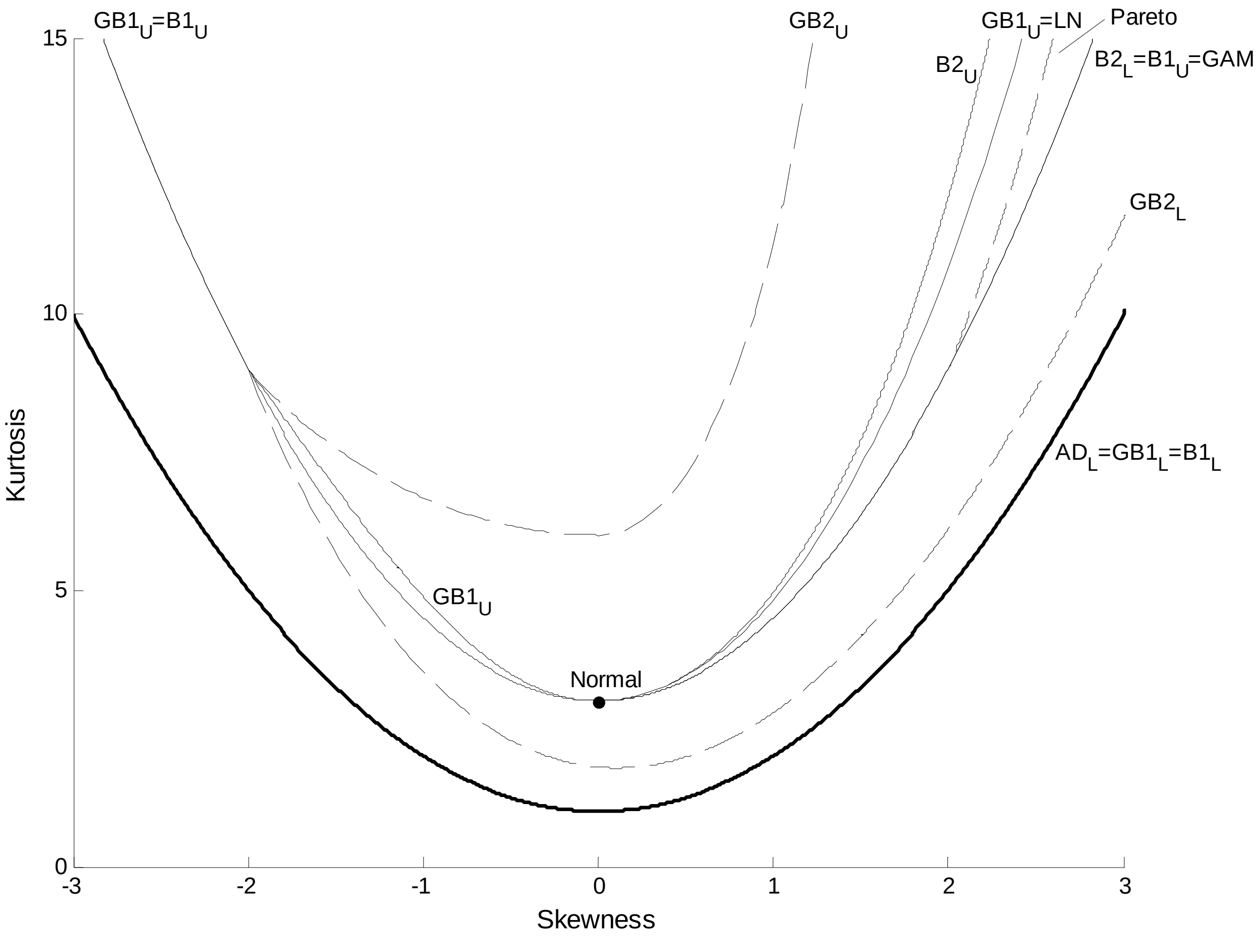
Skewness and kurtosis can also be expressed in terms of the distributional parameters. For some *pdf*s the permissible skewness-kurtosis combinations yield relatively simple expressions of the distributional parameters. For example, parametric expressions for feasible skewness and kurtosis combinations for the gamma can be written in terms of the distributional parameter *p* as $\gamma_1 = 2/\sqrt{p}$ and $\gamma_2 = 3 + 6/p$, which can be rewritten as $\gamma_2 = 3 + (3/2)\gamma_1{}^2$. For other distributions, tractable expressions for permissible kurtosis in terms of skewness have not been obtained, but parametric expressions for skewness and kurtosis are available. The Pareto and lognormal are two examples of distributions with fairly simple parametric representations (see the Appendix) that trace out feasible skewness-kurtosis combinations in the (γ_1, γ_2) plane. Similarly, the Weibull corresponds to a line in the (γ_1, γ_2) plane.

For the three-parameter distributions, the feasible skewness-kurtosis combinations correspond to two-dimensional regions (also referred to as spaces) in the (γ_1, γ_2) plane, which are defined by upper (U) and lower (L) bounds. Rodriguez (1977) explores feasible skewness-kurtosis combinations for the SM distribution. Tadikamalla (1980) derives the upper and lower bounds defining the Dagum skewness-kurtosis space and demonstrates that it includes the SM space as a proper subset. Vargo, Pasupathy, and Leemis (2010) summarize the skewness-kurtosis space corresponding to a number of distributions (including the LN, B1, B2, GG, GAM, and SM) and provide expressions for bounding curves for some of the distributions. Pearson (1916) demonstrates that $\gamma_2 \geq (\gamma_1)^2 + 1$ for all distributions. This inequality gives the lower bound for any empirical or theoretical distribution, and we refer to it as AD. Klaasen, et al. (2000) show that $\gamma_2 \geq (\gamma_1)^2 + 189/125$ defines a lower bound for unimodal distributions.

In the Appendix, we give upper and lower skewness-kurtosis bounds for the GB1 and GB2, summarize and expand on previously reported results, and provide explicit expressions for the bounding curves.

Figure 1 provides a graphical representation of the GB1, GB2, B1, B2, gamma, Pareto, lognormal, and normal skewness-kurtosis spaces. The normal corresponds to the point (0,3) in the (γ_1, γ_2) plane. The B1 and GB1 share the same skewness-kurtosis lower bound (represented in the figure as $B1_L$ and $GB1_L$, with bounds for other distributions labeled similarly); the lower bound for all distributions is AD_L. The gamma curve provides the upper bound for the B1 skewness-kurtosis space and the lower bound for the B2 space. The B2 allows for only positive skewness values, with the lower and upper bounds originating at (0,3). The GB2 skewness values are always greater than -2. For skewness values greater than -2, the GB2 lower bound is above the GB1 lower bound; however, the GB2 upper bound lies above the GB1 upper bound. While not obvious from the figure, the Pareto curve is contained in the B2 and GB2 spaces, but it lies above the upper bound for the more general GB1 distribution for skewness values exceeding 3.5. This is possible since the Pareto is a special case of the GB1 with $a = -1$, whereas the GB1, GB2, and their special cases correspond to $a \geq 0$. The Pareto also has a vertical asymptote at skewness of 7.07.

Figure 1. Skewness-kurtosis spaces for GB1, GB2, B1, B2, gamma, Pareto, lognormal, and normal

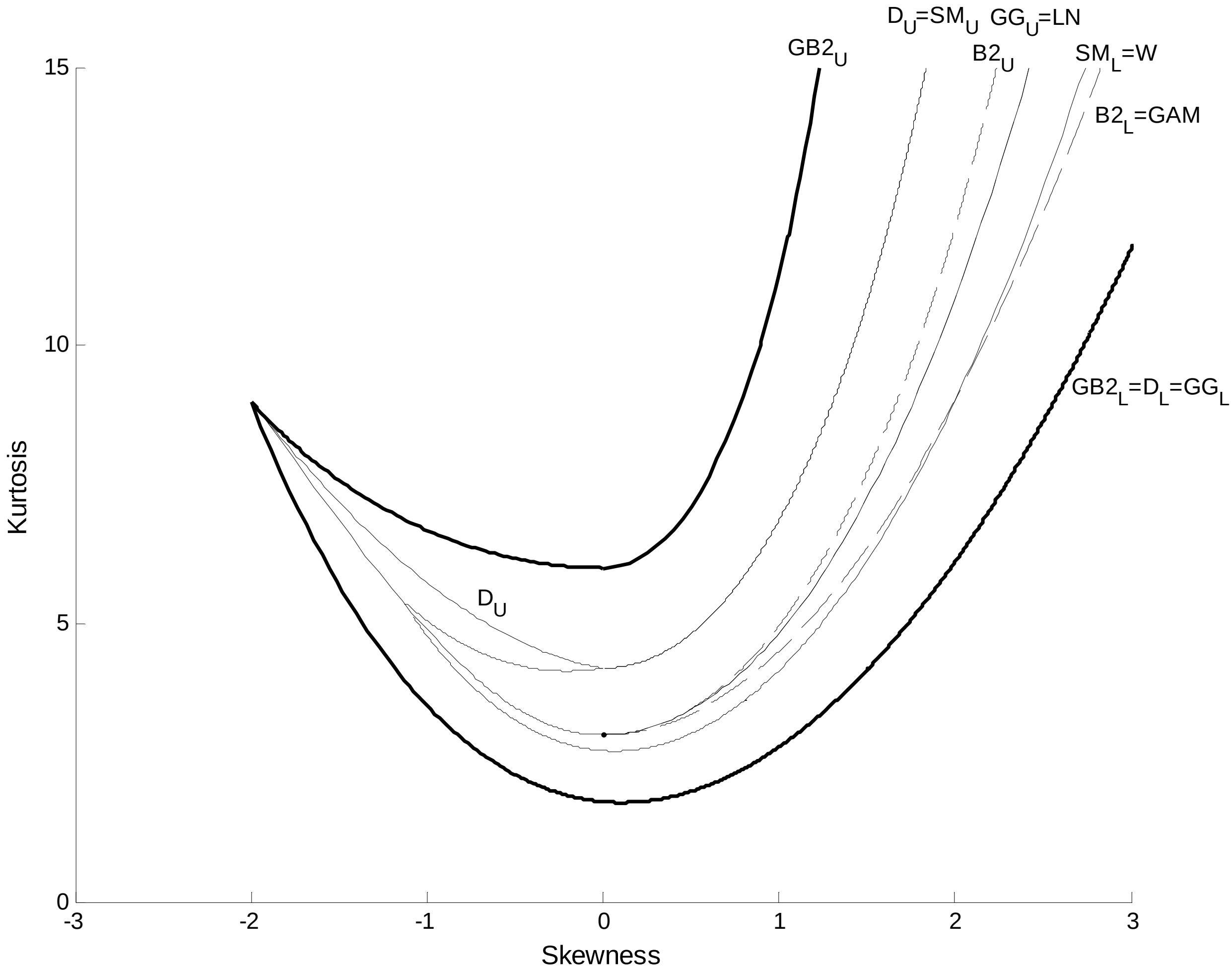


Note: The "L" subscript represents the lower bound and the "U" subscript represents the upper bound for the respective distribution's skewness-kurtosis space.

As most studies of income distributions employ special cases of the GB2 distribution, it is instructive to focus on the skewness-kurtosis spaces for the GB2 and its special cases, which are depicted in Figure 2. The GB2 upper bound lies above all of the upper bounds of its special and limiting cases. The Dagum and Singh-Maddala distributions share the same upper bound for positive skewness but differ slightly for negative skewness; the Dagum lower bound, however, lies below the Singh-Maddala lower bound (given by the Weibull skewness-kurtosis curve). The Dagum skewness-kurtosis space includes the

SM space as a proper subset, which helps explain why the Dagum distribution often provides a better fit than the Singh-Maddala distribution. The upper bound for the GG corresponds to the LN curve for positive skewness. The GB2, Dagum/SM, and B2 upper bounds have vertical asymptotes at skewness values of 2.30, 4.28, and 5.66, respectively. The generalized gamma, Dagum, and GB2 all share the same lower bound, which lies above the all distribution lower bound. Not surprisingly, the GB2 skewness-kurtosis space includes the spaces for all of its limiting and special cases.

Figure 2. Skewness-kurtosis spaces for GB2, Dagum, Singh-Maddala, B2, generalized gamma, lognormal, Weibull, and gamma



Note: The "L" subscript represents the lower bound and the "U" subscript represents the upper bound for the respective distribution's skewness-kurtosis space.

Table 1 reports upper and lower bounds (which define feasible skewness-kurtosis combinations) for the B1, B2, GG, Dagum, Singh-Maddala, GB1, and GB2. These bounds were used to construct Figures 1 and 2 and can assist researchers in selecting an appropriate distribution. Some of the distributions, such as the B2 and GG, have bounds that involve the skewness and kurtosis equations for other *pdf*s, such as the power function (PF), inverse gamma (IGAM), and log gamma (LGAM). Other distributions, such as the Dagum and SM, have bounds that are limiting or special cases of their own skewness and kurtosis equations. It is also worth noting that many bounds are segmented into two sections—one for the positive skewness part of the skewness-kurtosis plane and one for the negative skewness part. The Appendix includes skewness and kurtosis equations for all the distributions considered, a summary of the definitions and related properties of the additional *pdf*s mentioned above, and equations for the different skewness-kurtosis bounds.

To illustrate the interpretation of the results in Table 1, consider the B1 distribution. The possible combinations of (γ_1, γ_2) that can be modeled by the B1 are defined by the region bounded below by the all distribution lower bound ($\gamma_2 = \gamma_1^2 + 1$) and bounded above by the gamma skewness-kurtosis curve and its mirror image ($\gamma_2 = 3 + (3/2)\gamma_1^2$ for γ_1 real).

Table 1. Bounds for skewness-kurtosis spaces

pdf	Lower bound	Upper bound
B1	$(\gamma_1, \gamma_2)_{AD}$	$(\gamma_1, \gamma_2)_{GAM}$ and its mirror image
B2	$(\gamma_1, \gamma_2)_{GAM}$	$(\gamma_1, \gamma_2)_{IGAM}$
GG	$(\gamma_1, \gamma_2)_{PF}$	Negative skewness: $(\gamma_1, \gamma_2)_{LGAM}$ Positive skewness: $(\gamma_1, \gamma_2)_{LN}$
Dagum	$(\gamma_1, \gamma_2)_{PF}$	Negative skewness: $\lim_{a\to\infty}(\gamma_1, \gamma_2)_{DAGUM}$ Positive skewness: $(\gamma_1, \gamma_2)_{DAGUM}$ with p=1
Singh-Maddala	$(\gamma_1, \gamma_2)_{Weibull}$	Negative skewness: $\lim_{a\to\infty}(\gamma_1, \gamma_2)_{SM}$ Positive skewness: $(\gamma_1, \gamma_2)_{SM}$ with q=1
GB1	$(\gamma_1, \gamma_2)_{AD}$	Negative skewness: Mirror image of $(\gamma_1, \gamma_2)_{GAM}$ to the point (-2,9) and then $(\gamma_1, \gamma_2)_{LGAM}$ from (-2,9) to (0,3) Positive skewness: $(\gamma_1, \gamma_2)_{LN}$
GB2	$(\gamma_1, \gamma_2)_{PF}$	Negative skewness: $\lim_{p\to kq,\ a\to\infty}(\gamma_1, \gamma_2)_{GB2}$ Positive Skewness: See the Appendix

Note: Equations for and details about the various bounds are found in the Appendix.

We now consider an application of these models to actual income data and investigate which *pdf*s are sufficiently flexible to accommodate observed skewness and kurtosis values.

3. EMPIRICAL APPLICATION: LUXEMBOURG INCOME STUDY DATA

Household income data were obtained from the Luxembourg Income Study (LIS) database for 13 countries. The income measures we considered were earnings, gross income, and disposable income. Two time periods were used: Wave 5 of the survey (occurring in approximately 2000) and Wave 6 (occurring in approximately 2004). We looked at each country having data for each of the three income definitions for the time periods considered: Australia, Canada, Denmark, Finland, Germany, Israel, Norway, Poland, Sweden, Switzerland, Taiwan, United Kingdom, and the United States. An advantage of using the LIS data set is that the data from each country are uniformly formatted, especially with respect to the definition of income, thus facilitating inter-country comparisons. In all cases, income was measured in nominal local currency units. Because of government regulations and privacy laws, data on individual observations cannot be downloaded. Instead, we accessed the LIS microdata through their server to calculate the sample size, mean, variance, skewness, kurtosis, and Gini coefficient for each country, income definition, and year. Another advantage of using LIS data is that the income variables are continuous, not grouped, which makes the calculation of these measures more accurate.

Table 2 summarizes the definitions of income used in this study. Earnings measures income before taxes and transfer payments. Gross income measures income and transfer payments before taxes are withheld. Disposable income measures income after adjusting for taxes and transfer payments. We followed the recommendation of LIS group and used the weighted data, which can correct for non-sampling errors and sample bias. For additional details on the weighting procedures, see http://www.lisproject.org/techdoc.htm.

Table 2. Definitions of LIS income measures used

Earnings (income before taxes and transfer payments)

- Gross cash wage and salary income
- Farm self-employment income
- Non-farm self-employment income

Gross income (income before taxes and after transfer payments)

- Earnings
- Cash property income (includes cash interest, dividends, rents, annuities, royalties, etc.)
- Private occupational and other pensions
- Public sector occupational pensions
- Sickness benefits
- Occupational injury and disease benefits
- Disability benefits
- Maternity and other family leave benefits
- Military/veterans/war benefits
- Other social insurance benefits
- State old-age and survivors benefits
- Child/family benefits
- Unemployment compensation benefits
- Social assistance cash benefits
- Near-cash benefits
- Alimony/child support
- Regular private transfers
- Other cash income

Disposable income (income after taxes and transfer payments)

- Gross income
- Minus:
  - Mandatory contributions for self-employed (includes social security, unemployment, etc.)
  - Mandatory employee contributions
  - Income taxes

Source: http://www.lisproject.org/techdoc/sumincvar.xls.

Table 3 reports the sample size and distributional characteristics of interest for households reporting positive income. Not surprisingly, the income data exhibit positive skewness. There is considerable variation in the estimated values for standardized skewness and kurtosis. One questionable observation in the Sweden 2005 data had a value for interest and dividends that was nearly 200 times as large as the next reported value, which greatly affected skewness and kurtosis; hence, the observation was dropped for all our analyses (see notes to Table 3). As measured by the Gini coefficient, transfer payments and taxes result in a more egalitarian distribution in ten of the thirteen countries considered, with only Australia, Taiwan, and the UK having similar Gini coefficients for earnings and disposable income. In all cases, taxes applied to gross income resulted in smaller Gini coefficients; however, the decrease in Switzerland and Poland was quite small.

Table 3. Moments and Gini coefficients for LIS annual household income data

Country	Year	Definition	n	Mean	Std Dev	Skew	Kur	Gini
Australia	2001	Earnings	4510	59838	44358	2.902	20.90	0.360
		Gross income	6699	51288	45621	3.483	27.63	0.417
		Disposable income	6697	41786	31475	2.993	24.74	0.370
	2003	Earnings	6741	64936	49704	4.056	45.23	0.363
		Gross income	10087	55916	49276	4.457	54.28	0.412
		Disposable income	10086	44870	33037	3.209	32.66	0.365
Canada	2000	Earnings	22298	56295	57237	6.555	95.91	0.437
		Gross income	28936	56859	56367	7.688	139.02	0.413
		Disposable income	28902	43824	36684	5.989	90.43	0.373
	2004	Earnings	21594	61690	63207	6.741	117.54	0.451
		Gross income	27776	64030	61225	7.109	132.43	0.412
		Disposable income	27774	50777	41204	4.899	70.06	0.375
Denmark	2000	Earnings	59549	364336	267158	2.130	22.28	0.386
		Gross income	81916	367090	284729	7.934	241.12	0.360
		Disposable income	81904	243828	163848	9.142	389.48	0.322
	2004	Earnings	59824	402943	308210	2.978	38.06	0.392
		Gross income	83220	407209	310706	6.289	144.85	0.358
		Disposable income	83178	276847	187778	7.888	264.98	0.323
Finland	2000	Earnings	8916	184190	161674	29.34	3537.9	0.406
		Gross income	10420	199933	238812	83.84	14375	0.379
		Disposable income	10415	145051	145006	59.21	7455.6	0.338
	2004	Earnings	9358	35526	27538	1.723	11.70	0.410
		Gross income	11226	39312	45717	22.00	845.86	0.389
		Disposable income	11220	29286	31328	24.41	989.47	0.349
Germany	2000	Earnings	8051	74823	58503	3.247	32.45	0.386
		Gross income	10982	71752	60239	4.831	71.64	0.392
		Disposable income	10982	52682	39517	6.886	169.17	0.345
	2004	Earnings	8297	40009	31625	2.942	28.13	0.394
		Gross income	11290	39008	55806	71.045	7580.9	0.393
		Disposable income	11288	29239	49607	98.088	11873	0.346
Israel	2001	Earnings	4382	149884	152131	4.815	67.21	0.457
		Gross income	5768	145455	153356	6.164	95.35	0.447
		Disposable income	5768	111356	98716	8.651	213.7	0.384
	2005	Earnings	4762	147007	139566	3.939	43.15	0.447
		Gross income	6259	148009	156337	7.028	110.60	0.441
		Disposable income	6255	120024	116068	10.493	240.49	0.394
Norway	2000	Earnings	11474	376575	295371	4.747	123.53	0.392
		Gross income	12888	398639	402778	13.433	411.36	0.387
		Disposable income	12870	300334	312958	19.976	787.06	0.355
	2004	Earnings	10947	409276	337594	3.533	58.51	0.417
		Gross income	13116	474134	906537	62.127	5679.68	0.396
		Disposable income	13112	360090	849079	71.487	6964.48	0.370
Poland	1999	Earnings	24201	19542	23610	48.927	4995.8	0.428
		Gross income	31273	23141	21117	51.820	5890.7	0.325
		Disposable income	31253	20600	20024	60.742	7320.1	0.323
	2004	Earnings	23534	22952	24490	8.773	268.61	0.461
		Gross income	32032	26284	21695	9.196	311.27	0.354
		Disposable income	32027	24414	20412	10.281	384.77	0.352

Table 3—cont.

Country	Year	Definition	n	Mean	Std Dev	Skew	Kur	Gini
Sweden	2000	Earnings	10319	291187	265613	5.557	90.12	0.425
		Gross income	14473	316451	331621	32.828	2417.71	0.379
		Disposable income	14470	221536	211460	39.730	3285.33	0.347
	2005	Earnings	11950	334616	295442	5.988	174.03	0.420
		Gross income	16254	373877	291425	6.471	167.45	0.355
		Disposable income	16251	269439	181934	4.749	87.15	0.327
Switzer.	2000	Earnings	3015	93973	70423	11.226	363.34	0.328
		Gross income	3641	96491	76939	11.326	285.40	0.320
		Disposable income	3627	73143	60555	13.535	382.77	0.318
	2004	Earnings	2596	97945	60182	1.777	11.40	0.320
		Gross income	3267	98638	61917	4.091	57.03	0.305
		Disposable income	3245	73580	44750	3.169	32.17	0.301
Taiwan	2000	Earnings	12301	844894	571044	2.634	23.18	0.340
		Gross income	13801	934142	648958	2.658	20.27	0.345
		Disposable income	13800	893449	609208	2.562	19.60	0.341
	2005	Earnings	11903	832582	576645	1.981	12.99	0.357
		Gross income	13681	915325	681198	2.863	21.63	0.364
		Disposable income	13679	872331	640298	2.878	22.41	0.359
UK	1999	Earnings	15199	28716	29158	11.061	276.97	0.401
		Gross income	24944	24589	26516	11.428	327.73	0.426
		Disposable income	24830	19596	20515	14.157	471.86	0.398
	2004	Earnings	17025	35719	38471	11.299	258.90	0.408
		Gross income	27684	31028	34088	11.496	294.32	0.421
		Disposable income	27574	24800	27417	12.965	332.26	0.398
USA	2000	Earnings	39621	58669	58185	3.341	19.35	0.442
		Gross income	49304	57698	58508	3.438	20.84	0.450
		Disposable income	49294	44785	39883	3.511	23.91	0.404
	2004	Earnings	62366	63012	65661	4.066	29.05	0.448
		Gross income	75746	61877	64727	4.145	31.99	0.453
		Disposable income	75736	49534	46568	4.371	38.97	0.414

Notes: All values are expressed in units of national currency in use at time of data collection. One questionable observation in the Sweden 2005 data has reported interest and dividends (which is included in gross and disposable income but not in earnings) of 1,526,993,544, whereas the next largest value is 8,448,972 and the mean value of interest and dividends, after dropping the outlier, is 8,794. Thus, the observation has been dropped. Keeping the observation results in a mean, standard deviation, skewness, kurtosis, and Gini coefficient for gross income of 374,602, 1,094,773, 1,305, 1,831,523, and 0.357, respectively. It results in a mean, standard deviation, skewness, kurtosis, and Gini coefficient for disposable income of 269,945, 758,053, 1,333, 1,883,894, and 0.328, respectively.

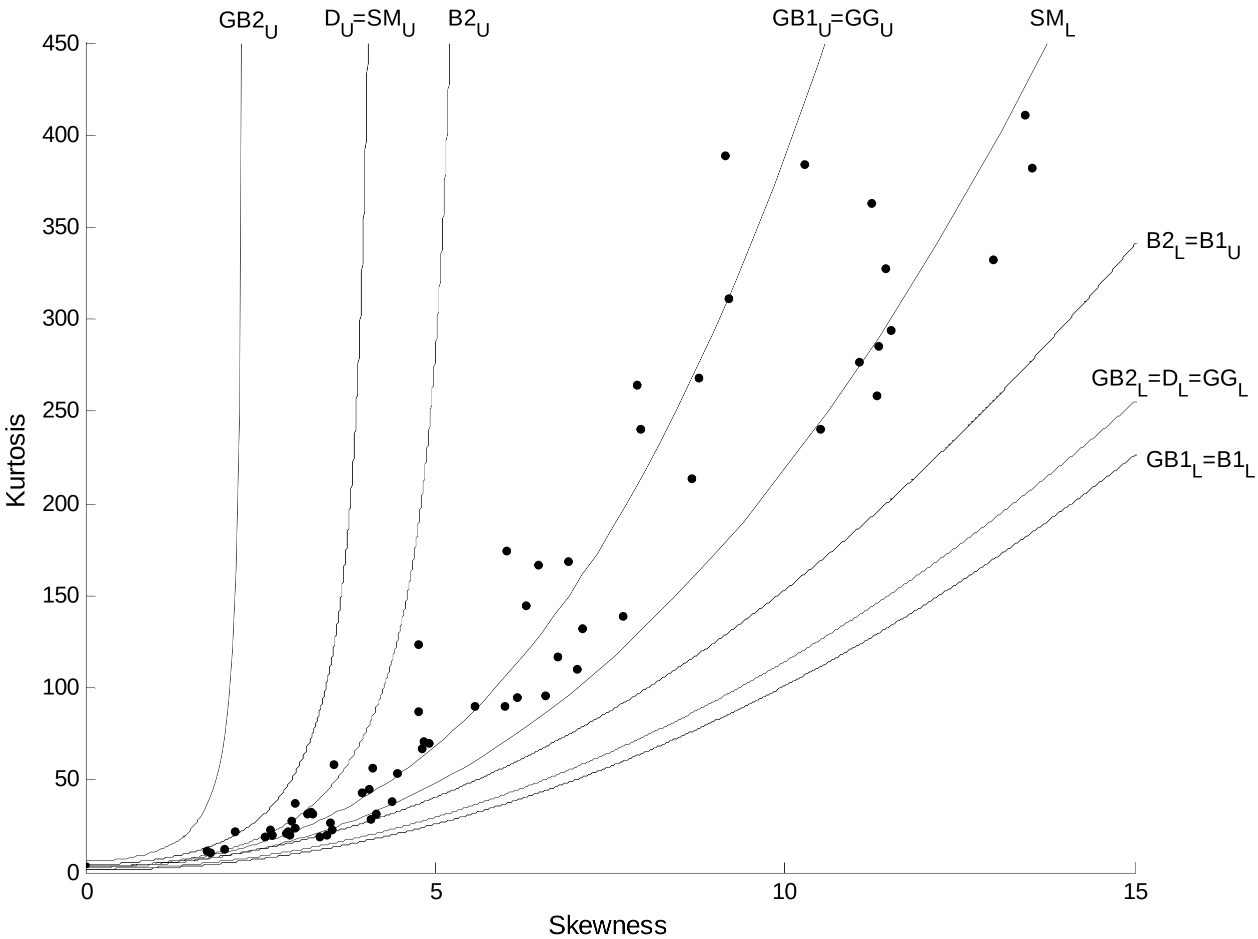
Table 4 reports the percent of the 78 cases (13 countries, 3 income definitions, 2 time periods) included in each of skewness-kurtosis spaces considered. While maximum likelihood estimators would not match sample and theoretical moments (as would method of moments estimators), these results shed light on the relative ability of the different *pdf*s to accommodate the observed distributional characteristics in the data considered. Only 3 of the 78 (3.85%) cases fall in the B1 skewness-kurtosis space; whereas, the B2 space includes 66 of the 78 (84.62%) cases considered. The Dagum clearly outperforms the Singh-Maddala distribution, accounting for all but one of the observations. Although the GB1 lower bound lies below the GG lower bound, the two distributions perform equally as well (none of the data points fall in the GB1's extended region).

Table 4. Percent of 78 data points included in each skewness-kurtosis space

pdf	% of data included in skewness-kurtosis space
GB2	100.00
Dagum	98.72
B2	84.62
SM	65.38
GB1	57.69
GG	57.69
B1	3.85

Figure 3 illustrates 62 of the 78 observed skewness and kurtosis combinations along with the skewness-kurtosis spaces considered. The scale of the figure was selected to facilitate distinguishing the different bounds, with 16 of the larger skewness-kurtosis data points being omitted.

Figure 3. Skewness-kurtosis data points and skewness-kurtosis spaces



4. SUMMARY AND CONCLUSIONS

The ability of some popular income distributions to model distributional characteristics was investigated. The GB1 and GB2 skewness-kurtosis spaces were evaluated, and prior results on the spaces for the Pareto, lognormal, gamma, Weibull, generalized gamma, Dagum, Singh-Maddala, and beta distributions were given, expanded on, and compared. Of the models considered, the GB2 allowed for the highest kurtosis values for positive skewness, which appears to be important in modeling the distribution of income. The skewness-kurtosis values observed for thirteen countries, three definitions of income, and two time periods were able to be modeled by the GB2. Of the three-parameter models, the Dagum performed the best and nearly as well as the more general GB2.

APPENDIX: SKEWNESS-KURTOSIS SPACES FOR SELECT DISTRIBUTIONS

A.1. Skewness and kurtosis equations

The evaluation of feasible skewness-kurtosis combinations is facilitated by analyzing a reparameterization of the standardized skewness and kurtosis given above:

$$\gamma_1 = \frac{\left(\lambda_0^2\lambda_3 - 3\lambda_0\lambda_1\lambda_2 + 2\lambda_1^3\right)}{\left(\lambda_0\lambda_2 - \lambda_1^2\right)^{3/2}}$$

$$\gamma_2 = \frac{\left(\lambda_0^3\lambda_4 - 4\lambda_0^2\lambda_1\lambda_3 + 6\lambda_0\lambda_1^2\lambda_2 - 3\lambda_1^4\right)}{\left(\lambda_0\lambda_2 - \lambda_1^2\right)^2}$$

where the $\lambda_h^{'}$s for different *pdf*s are presented in Table A.1. Expressions for the $\lambda_h^{'}$s correspond to the moment equations without the scale parameter and terms not involving *h* that cancel out in calculating the standardized skewness and kurtosis.

Table A.1. $\lambda_h^{'}$ s used to calculate skewness and kurtosis for different *pdf*s

pdf	λ_h (h=0,1,2,3,4)
GB1	$\Gamma(p+h/a)/\Gamma(p+q+h/a)$
GB2	$\Gamma(p+h/a)\Gamma(q-h/a)$
Dagum (Burr3)	$\Gamma(p+h/a)\Gamma(1-h/a)$
Singh-Maddala (Burr12)	$\Gamma(1+h/a)\Gamma(h-i/a)$
Generalized gamma	$\Gamma(p+h/a)$
Weibull	$\Gamma(1+h/a)$
Gamma	$\Gamma(p+h)$
Inverse gamma	$1/\Gamma(p-h)$
Log gamma	$\left(\frac{\beta}{\beta-h}\right)^p$
Power function	$1/(\theta+h)$

A.2. Skewness-kurtosis spaces

Pareto curve: The feasible combinations of (γ_1, γ_2) for the Pareto distribution can be expressed parametrically as $\gamma_1 = 2\left(\frac{p+1}{p\text{-}3}\right)\sqrt{1\text{-}2/p}$, $3<p$, and $\gamma_2 = \frac{3(p-2)(3p^2+p+2)}{p(p-3)(p-4)}$, $4<p$ (Kleiber and Kotz, 2003).

Power function curve: The feasible combinations of (γ_1, γ_2) for the power function can be expressed parametrically as $\gamma_1 = \frac{2(1-\theta)\sqrt{1+2/\theta}}{\theta+3}$ and $\gamma_2 = \frac{3(3\theta^2-\theta+2)(1+2/\theta)}{(\theta+3)(\theta+4)}$ (Kleiber and Kotz, 2003).[1]

Gamma curve: $\gamma_1 = \frac{2}{\sqrt{p}}$ and $\gamma_2 = 3 + \frac{6}{p}$ or $\gamma_2 = 3 + \frac{3}{2}\gamma_1^{\,2}$ (Kleiber and Kotz, 2003).

Inverse gamma curve: $IGAM(y;\beta,p) = GG(y;a=-1,\beta,p)$.

$\gamma_1 = \frac{4\sqrt{p\text{-}2}}{p\text{-}3}$ and $\gamma_2 = 3 + \frac{30p-66}{(p-3)(p-4)}$, $p>4$ (Vargo, et al., 2010).

Log gamma curve: $LGAM(y;\beta,p) = GAM(\ell n(y);\beta,p)/y$, $1<y$ (Kleiber and Kotz, 2003).

$\gamma_1 = \frac{\psi''(p)}{(\psi'(p))^{3/2}}$ and $\gamma_2 = \frac{\psi'''(p)}{(\psi'(p))^2}$ (Farebrother, 1990).

Lognormal curve: $\gamma_1 = \sqrt{e^{3\sigma^2} + 3e^{2\sigma^2} - 4}$ and $\gamma_2 = e^{4\sigma^2} + 2e^{3\sigma^2} + 3e^{2\sigma^2} - 3$ (Kleiber and Kotz, 2003).

Weibull curve: $\gamma_1 = \frac{\Gamma(3/a+1) - 3\Gamma(2/a+1)\Gamma(1/a+1) + 2\Gamma^3(1/a+1)}{\left(\Gamma(2/a+1) - \Gamma^2(1/a+1)\right)^{3/2}}$ and

[1] Kleiber and Kotz (2003) inadvertently indicate that the PF is a limiting case of the GG as $a \to 0$ with $p = \theta/a$; the correct limit is $a \to \infty$ with $p = \theta/a$.

$$\gamma_2 = \frac{\Gamma(4/a+1) - 4\Gamma(3/a+1)\Gamma(1/a+1) + 6\Gamma(2/a+1)\Gamma^2(1/a+1) - 3\Gamma^4(1/a+1)}{\left(\Gamma(2/a+1) - \Gamma^2(1/a+1)\right)^2}$$ (Rodriguez, 1977).

B1 space:

Upper bound: $\gamma_2 = 3 + \frac{3}{2}\gamma_1^2$, corresponding to the gamma for positive skewness and the mirror image for negative skewness (Vargo, et al., 2010). The limit of the B1 as $q \to \infty$ is a gamma *pdf*.

Lower bound: $\gamma_2 = 1 + \gamma_1^2$, the boundary for all distributions (Johnson and Kotz, 1995; Vargo, et al., 2010).

B2 space:

Upper bound: $\gamma_1 = \frac{4\sqrt{p-2}}{p-3}$ and $\gamma_2 = 3 + \frac{30p-66}{(p-3)(p-4)}$, $p > 4$, which corresponds to the inverse gamma family (Vargo, et al., 2010).

Lower bound: $\gamma_2 = 3 + \frac{3}{2}\gamma_1^2$, corresponding to the gamma, the limit of $(\gamma_1, \gamma_2)_{B2}$ as $q \to \infty$ (Vargo, et al., 2010).

Dagum space:

Upper bound:

- Negative skewness: $(\gamma_1, \gamma_2) = \lim_{a\to\infty}(\gamma_1, \gamma_2)_{DAGUM}$ (Tadikamalla, 1980), which gives $\gamma_1 = \frac{-6\sqrt{6}\left[\psi''(1) - \psi''(p)\right]}{\left(\pi^2 + 6\psi'(p)\right)^{3/2}}$ and

$$\gamma_2 = \frac{9\left(3\pi^4 + 20\pi^2\psi'(p) + 60\left(\psi'(p)\right)^2 + 20\psi'''(p)\right)}{5\left(\pi^2 + 6\psi'(p)\right)^2},\ 0<p<1$$

(obtained using Mathematica).

- Positive skewness: $(\gamma_1, \gamma_2)_{DAGUM}$ with p = 1 and a >4 (Tadikamalla, 1980).

Lower bound: $(\gamma_1, \gamma_2)_{DAGUM}$ with $p \to 0$ and a varying (Tadikamalla, 1980). This bound is numerically equivalent to the power function curve.

Singh-Maddala space:

Upper bound:

Negative skewness: $(\gamma_1, \gamma_2) = \lim_{a\to\infty} (\gamma_1, \gamma_2)_{SM}$ (Rodriguez, 1977), yielding

$$\gamma_1 = \frac{6\sqrt{6}\left[\psi''(1) - \psi''(q)\right]}{\left(\pi^2 + 6\psi'(q)\right)^{3/2}} \text{ and}$$

$$\gamma_2 = \frac{9\left(3\pi^4 + 20\pi^2\psi'(q) + 60\left(\psi'(q)\right)^2 + 20\psi'''(q)\right)}{5\left(\pi^2 + 6\psi'(q)\right)^2}, \ 1<q$$

(obtained using Mathematica).

Positive skewness: $(\gamma_1, \gamma_2) = (\gamma_1, \gamma_2)_{\mathrm{SM}}$ with q=1 and a>4 (Rodriguez, 1977).

Lower bound: Use the Weibull curve, which corresponds to the limit of the SM as $q \to \infty$ (Rodriguez, 1977), with the bounds intersecting at (γ_1, γ_2)= (-1.14, 5.35).

Generalized gamma space:

Bounds for the GG with $a \geq 0$ were found by using the GG skewness and kurtosis equations to optimize γ_2 subject to different values of γ_1 and to optimize γ_1 subject to different values of γ_2. Relationships with bounds for other distributions were investigated and various limits were evaluated. To confirm these results, skewness-kurtosis points were calculated for over a million different combinations of parameter values and then plotted. (See Johnson and Kotz (1972) for the GG bounds without the $a \geq 0$ restriction.)

Upper bound:

- Negative skewness: use the log gamma

$$\lim_{a\to\infty}(\gamma_1)_{GG} = \frac{\psi''(p)}{(\psi'(p))^{3/2}} = (\gamma_1)_{LGAM}$$

$$\lim_{a\to\infty}(\gamma_2)_{GG} = 3 + \frac{\psi'''(p)}{(\psi'(p))^2} = (\gamma_2)_{LGAM}$$

- Positive skewness: use $(\gamma_1, \gamma_2)_{LN} = \lim_{a\to 0}(\gamma_1, \gamma_2)_{GG}$

Lower bound: $\gamma_1 = 2\left(\frac{1-\theta}{\theta+3}\right)\sqrt{1+2/\theta}$ and $\gamma_2 = \frac{3(1+2/\theta)(3\theta^2-\theta+2)}{(\theta+3)(\theta+4)}$, $0<\theta$, which intersects the upper bound at (-2, 9). This corresponds to the power function curve and to $\lim(\gamma_1)_{GG}$ with $a\to\infty$ and $ap=\theta$.

GB1 space:

Bounds for the GB1 with $a\geq 0$ were found by using the GB1 skewness and kurtosis equations to optimize γ_2 subject to different values of γ_1 and to optimize γ_1 subject to different values of γ_2. Relationships with bounds for other distributions were investigated and various limits were evaluated. To confirm these results, skewness-kurtosis points were calculated for over a million different combinations of parameter values and then plotted.

Upper bound:

- Negative skewness: $(\gamma_1, \gamma_2) = \lim_{q\to\infty,\, a=1}(\gamma_1, \gamma_2)_{GB1} =$ mirror image of $(\gamma_1, \gamma_2)_{GAM}$ to the point (-2,9) and then $(\gamma_1, \gamma_2)_{LGAM}$ from (-2,9) to (0,3).
- Positive skewness: $(\gamma_1, \gamma_2) = \lim_{a\to 0,\, q\to\infty}(\gamma_1, \gamma_2)_{GB1} = (\gamma_1, \gamma_2)_{LN}$

Lower bound: $\gamma_2 = 1+\gamma_1^2$, the lower bound for all distributions.

GB2 space:

Bounds for the GB2 with $a\geq 0$ were found by using the GB2 skewness and kurtosis equations to optimize γ_2 subject to different values of γ_1 and to optimize γ_1 subject to different values of γ_2.

Relationships with bounds for other distributions were investigated, various limits were evaluated, and for the upper bound for positive skewness Padé approximations were used. To confirm these results, skewness-kurtosis points were calculated for over a million different combinations of parameter values and then plotted.

Upper bound:

- **Negative skewness:** $(\gamma_1, \gamma_2) = \lim_{p \to kq,\ a \to \infty} (\gamma_1, \gamma_2)_{GB2} =$

  $\left(\frac{2(k^3 - 1)}{(1+k^2)^{3/2}}, \frac{3(3+2k^2+3k^4)}{(1+k^2)^2} \right),\ 0 \le k$. (-2,9) is the left endpoint.

- **Positive skewness:** This part of the upper bound does not appear to be a limiting or special case of the GB2, and it also does not correspond to any of the other bounds examined in this paper. A Padé approximation for the bound is $\gamma_2 = (\gamma_0 x^5 + \gamma_1 x^4 + \gamma_2 x^3 + \gamma_3 x^2 + \gamma_4 x + \gamma_5) / (x^3 + \delta_1 x^2 + \delta_2 x + \delta_3)$

  where $x = \gamma_1$ and the estimated coefficients and 95% confidence intervals are given by

  $$\begin{aligned}
  \gamma_0 &= \text{-}3640 \ \ (\text{-}1.497e{+}007,\ 1.496e{+}007) \\
  \gamma_1 &= 4349 \ \ (\text{-}1.79e+007,\ 1.79e+007) \\
  \gamma_2 &= 6.776e+004 \ \ (\text{-}2.784e+008,\ 2.785e+008) \\
  \gamma_3 &= \text{-}5.444e+004 \ \ (\text{-}2.237e+008,\ 2.236e+008) \\
  \gamma_4 &= 2.033e+005 \ \ (\text{-}8.353e+008,\ 8.357e+008) \\
  \gamma_5 &= \text{-}1.203e+004 \ \ (\text{-}4.947e+007,\ 4.944e+007) \\
  \delta_1 &= \text{-}1.494e+004 \ \ (\text{-}6.141e+007,\ 6.138e+007) \\
  \delta_2 &= 3.526e+004 \ \ (\text{-}1.449e+008,\ 1.449e+008) \\
  \delta_3 &= \text{-}2065 \ \ (\text{-}8.487e+006,\ 8.483e+006)
  \end{aligned}$$

  One thousand optimized data points between skewness values of 0 and 2.3037 (the vertical asymptote) were used to fit the approximating equation, resulting in a RMSE of 0.0006 and a corresponding R^2 of 1.0000.

Lower bound: (γ_1, γ_2) corresponding to

$$\lim_{p\to 0,\ a\to\theta/p} (\gamma_1, \gamma_2)_{GB2} = \left(\frac{2(1-\theta)\sqrt{1+2/\theta}}{\theta+3}, \frac{3(1+2/\theta)(3\theta^2-\theta+2)}{(\theta+3)(\theta+4)} \right)$$

with the left endpoint (-2,9) being obtained from $\theta \to \infty, 0 < \theta,$ which is identical to the power function curve.

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