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Equity-regarding poverty measures: Differences in needs and the role of equivalence scales

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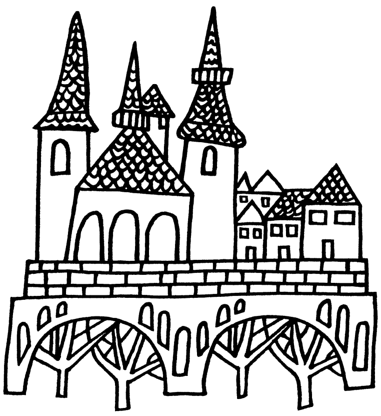

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Abstract

The paper investigates the definition of equity-regarding poverty measures when there are different household types in the population. It derives the implications of a between-type regressive transfer principle for poverty measures, for the choice of poverty lines and for the measurement of living standard. The role of equivalence scales which are popular in empirical work on poverty measurement is clarified.

Keywords: Poverty measures, differences in needs, principle of transfers, equivalence scales

JEL-codes: D63, I31, I32

1. Introduction[1]

Following Sen's (1976) seminal article it is generally accepted today that for the measurement of poverty the distribution of income among the poor matters. Sen has examined poverty in a homogeneous population comprising identical individuals. In this paper we consider a population consisting of households which may differ in type: Their size and composition can be different and they can have different needs. For a heterogeneous population household income is no longer a reliable base for poverty measurement. Instead we have to use an indicator of the household's living standard which depends on the household's income and its characteristics. The objective of the analysis below is to develop equity-regarding poverty measures based on the standard of living attained. An equity-regarding poverty measure has to satisfy a transfer principle requiring that a regressive transfer of income from a poorer household to a less poor household increases poverty – where poorness is measured by the household's living standard. This transfer principle incorporates the idea of equity and is a value judgment.

The impact of a transfer depends primarily on the definition of living standard, or more generally on the way differences in household type (and needs) are taken into account. The literature on poverty measurement offers various approaches to dealing with differences in needs: For instance in theoretical work, Keen (1992) chooses type-specific poverty lines, but does not adjust household incomes for needs. A simple possibility of defining living standard is to use equivalence scales which reflect a household's needs and to transform household incomes into equivalent incomes measured for a reference type.[2] Equivalent income is then a representation of living standard. Pyatt (1990) and Hagenaars (1987) follow this route and employ equivalence scales for deflating incomes. Hagenaars uses the number of individuals belonging to the household for weighting the individuals' contributions to poverty. On the other hand in Pyatt's paper these contributions are weighted by the corresponding equivalence scale values. The same variety of methods can also be found in empirical work, cf. e.g., Szulc (1995), de Vos and Zaidi (1997), Osberg (2000), Finnie and Sweetman (2003), and Hunter, Kennedy and Biddle (2004).

Given these various approaches, this paper aims at presenting a general and consistent method of defining equity-regarding poverty measures for a heterogeneous population. The analysis is based on a number of assumptions and ingredients. First, since in practice often only the distribution of household income and household composition can be observed, we a priori con-

[1] I thank Martin Duensing, an Editor, two anonymous referees, and in particular Peter Lambert for helpful comments and suggestions.

[2] An alternative is to use a dominance approach, see for instance Atkinson (1992) and Zheng (2000).

sider households. Second, we start with a family of poverty measures which are additively decomposable in the households' contributions to poverty. These contributions are measured by means of a deprivation function which describes the degree of poverty for each household. Third, as we have to define and to compare the living standard of different household types we introduce a general concept, the equivalent income function suggested by Donaldson and Pendakur (2004). It transforms household income into equivalent income. Equivalence scales represent a particular case of this concept. Finally, the between-type transfer principle is formulated in this framework. The principle considers transfers between different household types and postulates that a regressive transfer of income from a poorer (or less well-off) household to a less poor (or better-off) household increases poverty. Here the households' living standard is measured by means of equivalent income.

This setting allows us to derive the implications of the transfer principle for the measurement of poverty. It turns out that the transfer principle can only be satisfied if there is a close connection between the way living standards are measured on the one hand and the way the poverty measure and poverty lines are chosen on the other hand. The relationship between these concepts is precisely described and the class of equity-regarding measures fulfilling the transfer principle is completely characterized. Since equivalence scales are popular in empirical work, the role they play for (the form of) equity-regarding poverty measures is also discussed and clarified.

The paper contributes to the literature on the redistribution of income in heterogeneous populations. This literature mostly deals with this topic for the measurement of welfare or inequality.[3] This paper examines the issue for poverty measures. Pyatt (1990) has already made some remarks on poverty measurement in his discussion of social evaluation criteria. Ebert (2004) has considered the ethical measurement of inequality and poverty. It is based on social welfare functions. In Ebert (2008) the meaning of a between-type progressive transfer principle for social welfare functions is explored in an abstract framework which in principle can be used to derive poverty measures. Compared to these contributions the focus of this paper is different: The present analysis concentrates on poverty measures from the beginning and investigates the consequences of imposing the transfer principle for the structure of these measures, the choice of poverty lines and the way living standard is measured.

3 In a general framework between-type progressive transfers are considered in Hammond (1977), Ebert and Moyes (2003), Ebert (2004, 2007, 2008), and Shorrocks (2004). For the particular case that equivalence scales are used to define living standard, these transfers are examined in Glewwe (1991), Ebert (1997, 1999), Pyatt (1990), Decoster and Ooghe (2003), and Ooghe and Lambert (2006).

The paper is organized as follows: Section 2 describes the framework. Poverty measures and the concept of living standard are defined and the principle of between-type regressive transfers is introduced. In order to illustrate the concepts used an empirical example is examined in section 3. Section 4 provides a characterization of poverty measures satisfying the transfer principle and discusses its specific implications when equivalence scales are employed or the poverty measures possess particular properties. Section 5 concludes.

2. Framework

We consider a heterogeneous population consisting of $N \geq 2$ households which have different sizes, composition and/or needs. In order to concentrate on the topic of redistribution between different household *types*, in the theoretical analysis we take into account exactly[4] one household for each type and assume that household *i* has type *i* for $i = 1,\ldots,N$. No needs ranking is imposed. Denote household *i*'s money income by $X_i \in D$ and let the vector $\mathbf{X} = (X_1,\ldots,X_N) \in D^N$ be the corresponding income distribution where $D = \mathbb{R}_{++}$ or $D = \mathbb{R}$.

2.1 Poverty measures

At the beginning we suppose that there is a separate poverty line[5] $Z_i > 0$ for each household type *i* for $i = 1,...,N$. These are combined into a vector $\mathbf{Z} = (Z_1,...,Z_N) \in D^N$. Below, the poverty lines will be related and harmonized.

Up to a normalizing factor a poverty measure *P* is a function $P : D^N \times \mathbb{R}^N_{++} \to \mathbb{R}_+$ which is defined by

$$P(\mathbf{X},\mathbf{Z}) = \sum_{i=1}^{N} p^i(X_i, Z_i) \tag{1}$$

where $p^i : D \times \mathbb{R}_{++} \to \mathbb{R}_+$ is the (so-called) deprivation function of household type *i*, for $i = 1,...,N$. The function $p^i(X_i, Z_i)$ measures the deprivation or the contribution to poverty of household *i* with income X_i when the poverty line for household type *i* is equal to Z_i. A measure *P* defined in (1) is additively decomposable.

4 This assumption is dropped in the discussion of the empirical example in section 3.

5 We postulate that the poverty line is strictly positive in order to guarantee a subsistence level.

A poverty measure is called *regular* if each deprivation function $p^i\left(X_i, Z_i\right)$ (for $i=1,...,N$) satisfies some basic properties: (i) The function is continuous at all incomes $X_i \in D$ and twice continuously differentiable if $X_i \neq Z_i$. (ii) It is strictly positive for $X_i < Z_i$ and zero for $X_i \geq Z_i$ (when household *i* is nonpoor). (iii) Deprivation is strictly decreasing and marginal deprivation is (absolutely) non-increasing in income if $X_i < Z_i$, i.e. $p_X^i\left(X_i, Z_i\right) < 0$ and $p_{XX}^i\left(X_i, Z_i\right) \geq 0$ for $X_i < Z_i$ where p_X^i and p_{XX}^i denote partial derivatives with respect to X_i. A regular poverty measure is called *smooth* if the deprivation functions p^i are also continuously differentiable at the poverty lines Z_i, for $i=1,\ldots,N$.

A regular poverty measure $P(\mathbf{X}, \mathbf{Z})$ is nonnegative by definition and is decomposable and subgroup-consistent by construction (i.e. if poverty in any subgroup increases so does overall poverty). Convexity of the deprivation functions is a necessary property: It implies that regressive transfers between poor households of the *same* type increase poverty (if there is more than one household of the same type in the population considered).

A poverty measure *P* is called a *relative* measure if $D=\mathbb{R}_{++}$ and if there are functions $g^i: D \rightarrow \mathbb{R}_+$ for $i=1,...,N$ such that

$$p^i\left(X_i, Z_i\right) = g^i\left(X_i / Z_i\right). \tag{2}$$

P is called an *absolute* measure if $D=\mathbb{R}$ and if there are functions $g^i: D \rightarrow \mathbb{R}_+$ for $i=1,...,N$ such that $p^i\left(X_i, Z_i\right) = g^i\left(Z_i - X_i\right)$.

If *P* is a relative measure we obtain $P(\mathbf{X}, \mathbf{Z}) = P(\lambda\mathbf{X}, \lambda\mathbf{Z})$ for $\lambda > 0$ and, if it is an absolute measure, we get $P(\mathbf{X}, \mathbf{Z}) = P\left(\mathbf{X}+\varepsilon\mathbf{1}_N, \mathbf{Z}+\varepsilon\mathbf{1}_N\right)$ for $\varepsilon > 0$ where $\mathbf{1}_N$ denotes a vector containing *N* ones. The measure is then invariant with respect to equal proportional and, respectively, absolute changes of all incomes and poverty lines.

Many poverty measures for a homogeneous population[6] are consistent with the framework described and can be generalized appropriately, e.g. the poverty gap $G(\mathbf{X}, Z) = \frac{1}{N}\sum_{X_i \leq Z}\left(Z - X_i\right)$ $\left[p(X, Z) = Z - X \text{ for } X \leq Z\right]$, the poverty ratio

[6] Cf. e.g. Zheng (1997). Here the vector **Z** boils down to the poverty line *Z* because of the homogeneity of the population.

$I(\mathbf{X},Z)=\frac{1}{N}\sum_{X_i\le Z}(1-X_i/Z)$ $\left[p(X,Z)=(Z-X)/Z \text{ for } X\le Z\right]$, the ethical measures $H_\gamma(\mathbf{X},Z)=\frac{1}{N}\sum_{X_i\le Z}\left(1-(X_i/Z)^\gamma\right)$, $0<\gamma<1$, $\left[p(X,Z)=1-(X/Z)^\gamma\right]$ (Chakravarty (1983)), and the Foster, Greer, and Thorbecke measures $P_\alpha(\mathbf{X},Z)=\frac{1}{N}\sum_{X_i\le Z}(1-X_i/Z)^\alpha, \alpha\ge 2$ $\left[p(X,Z)=((Z-X)/Z)^\alpha \text{ for } X\le Z\right]$ (Foster, Greer, Thorbecke (1984); see Ebert and Moyes (2002) for a characterization). The measures *G*, *I* and H_γ generate regular poverty measures and the measure P_α generates a smooth poverty measure.

2.2 Living standard

Now we turn to a discussion of living standards and needs. Since the households considered differ in needs their income levels cannot be employed directly to infer and to compare their living standards. In order to not restrict the analysis a priori, we therefore use the general concept of an equivalent income function for the definition of living standards (see Donaldson and Pendakur (2004) and Ebert (2000, 2004)). We assume that household 1 contains a single adult and that type 1 is the reference type. For the following we assume that there is no information about the distribution of income within households and therefore we shall suppose that all members belonging to a household attain the same standard of living. The basic idea is to measure the living standard of household type *i* in terms of the equivalent income of a single adult. An equivalent income function **E** is given by a vector of functions[7] $(E_1,\dots,E_N)$ such that $E_i : D\to D$ for $i=1,\dots,N$. Then $E_i(X_i)$ is equal to the (equivalent) income a *single* adult needs in order to be as well off as a member of household *i* which possesses income X_i. Living standard will be identified with the respective equivalent income. Thus we define: household *h* is (weakly) better off than household *i* if and only if $E_h(X_h)\ge E_i(X_i)$. It should be mentioned that in general equivalent income is not identical to a household's average income since there are economies of size (a household needs only one telephone, one refrigerator etc.).

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7 See Ebert (2000) for a derivation of equivalent income functions/equivalizing procedures.

We impose the following conditions on an equivalent income function: (i) $E_i\left(X_i\right)$ is continuously differentiable[8] for $X_i \in D$ and for $1 \le i \le N$. (ii) A single adult's income has not to be transformed since a single adult is the reference type ($E_1\left(X_1\right) = X_1$ for $X_1 \in D$). (iii) Living standard is strictly increasing in household income ($E_i\left(X_i\right)$ is strictly increasing in $X_i \in D$ for $1 \le i \le N$). (iv) Arbitrary living standards of all household types can always be compared ($E_i\left(D\right) = D$ for $1 \le i \le N$). Therefore it is possible to choose any household type as reference type (this property is important as there are no a priori arguments for choosing a particular reference type; we chose type 1 as reference above, merely for convenience).

In this framework e.g. relative [absolute] equivalence scales are represented by an equivalent income function having the form $E_i\left(X_i\right) = X_i / m_i$ $\left[E_i\left(X_i\right) = X_i - a_i\right]$ for $m_i \in \mathbb{R}_{++}$ $\left[a_i \in \mathbb{R}\right]$ $i = 1, ..., N$ and $m_1 = 1$ $\left[a_1 = 0\right]$. But, of course, the concept of **E** allows to define and to employ more general functional forms (see the Example presented below).

2.3 Between-type transfers

If income is redistributed between households of *different* types, needs matter and it is not clear under what conditions a regular poverty measure increases or decreases. Income levels do not allow us to decide which household is better off or worse off. For this kind of comparison living standards or equivalent incomes have to be used. We introduce a generalized version of the Pigou-Dalton principle of transfers:[9]

Definition (Regressive transfer)

Y *is obtained from* **X** *by a* between-type regressive transfer (BTRT) with respect to **E** and **Z** *if there are* $i, j \quad \left(1 \le i, j \le N, \quad i \ne j\right)$ *and* $\varepsilon > 0$ *such that*

$$Y_i = X_i - \varepsilon, \quad Y_j = X_j + \varepsilon, \; Y_k = X_k \; \textit{otherwise}$$

$$\textit{and} \quad E_i\left(Y_i\right) < E_i\left(X_i\right) \le E_j\left(X_j\right) < E_j\left(Y_j\right) \le \min\left\{E_i\left(Z_i\right), E_j\left(Z_j\right)\right\}.$$

In the definition of a BTRT it is assumed that the households affected by the transfer have to be poor before and after the redistribution of income according to both poverty lines: 'poor' is

8 Differentiability is not a core property of an equivalent income function, but simplifies things below (cf. the regularity condition for the deprivation functions!).

9 Cf. the references in footnote 3.

measured by the minimum of $E_i\left(Z_i\right)$ and $E_j\left(Z_j\right)$, i.e., by the living standards which have to be attained in order to become non-poor. It is possible that income is redistributed from a needier household type to a less needy one, e.g. from a poor couple to a less poor single adult. Thus household type is not the relevant criterion, it is the living standard measured by means of the equivalent income function **E** and the poverty lines **Z** which are decisive for the direction of the redistribution of income.

The corresponding transfer principle (depending on **E** and **Z**) is given by

Between-type Regressive Transfer Principle BTRT(E,Z)

$P\left(\mathbf{Y},\mathbf{Z}\right)\geq P\left(\mathbf{X},\mathbf{Z}\right)$ *whenever* **Y** *is obtained from* **X** *by a between-type regressive transfer with respect to* **E** *and* **Z**.

This generalization of the Pigou-Dalton principle seems to be natural in the heterogeneous framework. A poverty measure satisfying BTRT(**E**,**Z**) is called *equity-regarding*.

3. Application and discussion of concepts

In this section we illustrate the concepts introduced above and apply them to the measurement of poverty in Germany for the year 2000. The data are taken from the Luxembourg Income Study database (LIS (2008)). They originate from a sample of micro-data, which comes from the German Socio-Economic Panel (GSOEP) and contains 10,985 households. We utilize the LIS-weighted version of these data and the standardization[10] usually applied to LIS data bases.

For our investigation we consider the households' disposable income (measured in Deutsch Mark) and size (number of household members).

In order to describe the framework precisely a number of points have to be discussed:

(a) We have to define living standard and therefore adopt an equivalent income function. Numbering the household types by the number of household members, we employ the equivalence scales $m_i = i^{1/2}$, i.e., the functions $E_i\left(X_i\right) = X_i / m_i$ for $i = 1,\dots,N$. These particular scale values are often used in practice (cf. OECD (1995)).

10 E.g. top and bottom coding is applied. Missing values and zero incomes are excluded. See LIS Key Figures (2008) for all details.

(b) For poverty measurement we need poverty lines Z_i, for $i = 1, \ldots, N$. Since we want to use the same poverty standard for all household types we define $Z_i = m_i Z$ for $i = 1, \ldots, N$ where Z denotes the poverty line for a single adult. We set Z equal to 60 % of the median living standard. Then we obtain $Z = 20{,}483$ DM (cf. LIS Key Figures (2008)).

(c) When working with the empirical data we have to admit many households of each type. Furthermore, we have to choose a poverty measure for illustration. We use[11] the Foster, Greer, Thorbecke measure

$$P^*(\mathbf{X}, \mathbf{Z}) = \frac{1}{\Sigma w_k} \sum_{i=1}^{N} \sum_{j=1}^{n_i} w_i \left(\frac{Z - X_j^i / m_i}{Z} \right)^2 \tag{3}$$

where $\mathbf{X}^i = \left(X_1^i, \ldots, X_{n_i}^i \right)$ denotes the income vector of n_i type *i*-households (for $i = 1, \ldots, N$) and $\mathbf{X} = \left(\mathbf{X}^1, \ldots, \mathbf{X}^N \right)$. In this case the deprivation function for a household of type *i* is given by $p^i\left(X_i, Z_i\right) = w_i \left(\left(1 - \left(X_i / m_i\right)\right) / Z \right)^2$, i.e., it depends on a strictly positive weight w_i.

(d) We employ three different kinds of weighting which reflect the approaches discussed in the introduction:

Method (i): Households receive the *same* weight ($w_1 = \ldots = w_N$; cf. Keen (1992)).

Method (ii): Households are weighted according to the *number of household members* ($w_i = i$ for $i = 1, \ldots, N$; cf. Hagenaars (1987)).

Method (iii): Households are weighted according to the *equivalence scale value* ($w_i = m_i$ for $i = 1, \ldots, N$; cf. Pyatt (1990)).

Given this framework we are now able to investigate between-type regressive transfers. In order to obtain a visible effect on the poverty measure we have to redistribute income among *groups* of households in view of the great number of households. Then things are a bit more complicated than described in subsection 2.3: Suppose that we want to redistribute[12] income from *r* households of type *i* to *s* households of type *j*. If we take the amount ε from each type *i*-household we collect the total amount $r\varepsilon$ which has to be distributed equally among the *s* type *j*-households: Then each household of type *j* receives the amount $\varepsilon' = r\varepsilon / s$ and the

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[11] In order to be precise we distinguish between P and P^*: In the definition of P we have $n_1 = \ldots n_N = 1$. P^* is normalized.

[12] Here we assume that $\max X_k^i / m_i \leq \min X_l^j / m_j$ for the households concerned.

corresponding changes in living standard are described by $X_k^i / m_i \to \left(X_k^i - \varepsilon \right) / m_i$ and, respectively, $X_l^j / m_j \to \left(X_l^j + \varepsilon' \right) / m_j$. It has to be emphasized that such a regressive transfer between the two groups can be decomposed into a finite number of separate regressive transfers between single households.

It turns out[13] that in general poverty is increased by such between-type regressive transfers – independently of the method of weighting chosen[14], i.e., the poverty measure reacts as required by the transfer principle. But it is also possible to find transfers which violate the BTRT(**E**,**Z**) principle in some cases. We present two particular transfers:

Transfer A: We transfer the amount ε = 3,000 DM from poor two-person households to less poor one-person households.

Transfer B: We transfer the amount ε = 3,000 DM from poor two-person households to less poor three-person households.

The details are specified in Table 1:

	Transfer A		Transfer B	
	Type of donors	Type of receivers	Type of donors	Type of receivers
Household type	2	1	2	3
Living standard	9,000-10,000 DM	10,000-11,000 DM	4,000-5,000 DM	5,000-6,000 DM
Number of households in the LIS-weighted dataset	31,737	100,549	6,764	29,997

Table 1: Definition of transfers.
Source: Luxembourg Income Study version of the 2000 German Socio-Economic Panel: DEOOH Release 1.

Both transfers are regressive. Their impact on the poverty measure P^* is given by Table 2:

13 All computations have been performed in SPSS.

14 Since in this case the result is as one would expect we do not provide examples. The circumstances under which the transfer principle is violated are discussed below.

Method	Status quo	Effect of transfer A	Effect of transfer B
(i)	1276.9722	1276.1396	1277.5387
(ii)	1186.2401	1189.5249	1186.1449
(iii)	1234.3091	1235.6823	1234.6328

Table 2: Multiple of the poverty measure defined in (3): $10,000 \cdot P^*$.
Source: Luxembourg Income Study version of the 2000 German Socio-Economic Panel: DEOOH Release 1.

The poverty measure based on method (i) [method (ii)] violates the transfer principle for transfer A [transfer B]. The principle is satisfied if method (iii) is employed. The effects on the poverty measure of the transfers considered are relatively modest, but the share of poor households involved is very small (there are 36,288,149 households in the LIS-weighted data set).

To reveal the reason for these violations we disregard the fact that we have performed a transfer between *groups* of households and discuss a sufficient condition for the satisfaction of the BTRT(**E**,**Z**) principle between *two* households. Under the assumption that we redistribute the amount ε from an *i*-person household with income X_k^i to a *j*-person household with income X_l^j, the BTRT(**E**,**Z**) principle is satisfied by the poverty measure P^* defined in (3) if the contribution to poverty of these households

$$w_i \left(\frac{Z - \left(X_k^i - \varepsilon \right) / m_i}{Z} \right)^2 + w_j \left(\frac{Z - \left(X_l^j + \varepsilon \right) / m_j}{Z} \right)^2$$

is decreasing in ε. Forming the derivative of this expression and considering the specific (extreme) case that $X_k^i / m_i = X_l^j / m_j$ and $\varepsilon = 0$ we get the condition

$$\frac{w_i}{m_i} \geq \frac{w_j}{m_j}. \tag{4}$$

In this framework (4) it is necessary and – given our assumptions – even sufficient for the BTRT(**E**,**Z**) principle. Thus in this example the relationship between the weights and the equivalence scales is decisive for the outcome of a between-type regressive transfer.

Now one can easily see that condition (4) can be violated if method (i) or (ii) is employed:

For method (i) (4) is not satisfied if $m_i > m_j$ (since $w_i = w_j$ by assumption) which is the case for transfer A. For method (ii) ($w_i = i, w_j = j$) (4) is violated if $i / i^{1/2} < j / j^{1/2}$ or $i < j$, a condition fulfilled by transfer B. Finally, it is impossible to violate (4) for method (iii) as it boils down to $1 \geq 1$ (since $w_i = m_i$, $w_j = m_j$), a condition which is always satisfied.

Indeed, we will prove below that – when equivalence scales are used – the BTRT(**E**,**Z**) principle is satisfied *if and only if* the weights are equal to the corresponding scale values. Given this result the violations of the transfer principle can be interpreted intuitively: If method (i) is employed, the weights for multiperson households are too small. As a consequence, for transfer A the increase in group *i*'s contribution to poverty is not large enough to compensate the decrease implied for group *j*. Similarly, for method (ii) the weights of multiperson households are too large. Therefore the decrease in group *j*'s contribution to poverty is too large for transfer B. Thus the negative effect is dominating. In practice – and in the example – things are, of course, more complicated since also the *distribution* of household income within the groups affected by a transfer has an impact on the (change of) the poverty measure.

In summary, the example demonstrates that for the particular poverty measure P^* the choice of the weights is relevant if the BTRT(**E**,**Z**) principle is to be satisfied. In the following we investigate the general structure of equity-regarding poverty measures in the framework introduced in section 2.

4. Equity-regarding poverty measures

Now we derive the implications of the between-type regressive transfer principle for the class of poverty measures defined above.

4.1 Characterization

At first we characterize the transfer principle by some conditions. Afterwards we discuss the role of equivalence scales for equity-regarding poverty measures. The meaning and the implications of the transfer principle are described by[15]

Proposition 1

Let P be a regular [*smooth*] *poverty measure,* **Z** *a vector of poverty lines, and* **E** *an equivalent income function. Then the following statements are equivalent:*

[15] All proofs have been relegated to Appendix 1.

(a) *P satisfies BTRT(**E**,**Z**)*

(b) $p_X^i\left(X_i, Z_i\right) = p_X^j\left(X_j, Z_j\right)$ *for all* $X_i, X_j \in D$ *such that* $E_i\left(X_i\right) = E_j\left(X_j\right) \le \min\left\{E_i\left(Z_i\right), E_j\left(Z_j\right)\right\}$ *and all* $i, j = 1,\dots,N, \quad i \ne j$.

$$[\; p_X^i\left(X_i, Z_i\right) = p_X^1\left(E_i\left(X_i\right), Z_1\right) \text{ for all } X_i \in D \text{ and } E_i\left(Z_i\right) = Z_1 \text{ for } i = 2,\dots,N \;] \tag{5}$$

Proposition 1 is the fundamental result of this paper. BTRT(**E**,**Z**) imposes (and is equivalent to) a condition in which the poverty measure, the equivalent income function and the poverty lines are involved. (b) demonstrates that the marginal deprivation functions of different types have to be related: Since a between-type regressive transfer has to increase poverty, the donor's marginal deprivation has to be (absolutely) greater than the receiver's marginal deprivation. As for a regular measure marginal deprivation is always (absolutely) decreasing in income and as the principle BTRT(**E**,**Z**) depends on the living standards attained (and not on the household types involved) the marginal deprivation functions of all types have to be identical if the living standards are the same.[16] Furthermore, for smooth measures a common poverty line $E_i\left(Z_i\right) = Z_1$ for $i = 2,\dots,N$ is required, i.e., the living standard corresponding to the poverty line has to be independent of the type of household and all household types have to be treated equally. Thus BTRT(**E**,**Z**) guarantees horizontal equity (measured with respect to living standard). Proposition 1 demonstrates that in our framework the transfer principle can only be satisfied if the three ingredients of equity-regarding poverty measures – the measure *P*, the equivalent income function **E**, and the poverty lines $Z_1,\dots,Z_N$ – are connected and consistent.

In the following we restrict the setting by imposing further conditions and investigate the principle BTRT(**E**,**Z**) in more specific environments. We consider three scenarios: To begin with we assume that equivalent income can a priori be described by means of (relative) equivalence scales. Afterwards we impose two properties on the equity-regarding poverty measure which imply that living standards have to be compared by (relative or absolute) equivalence scales.

(a) **Employing equivalence scales**

It is shown below in subsection 4.2 that in principle arbitrary equivalent income functions can be used for the definition (or construction) of equity-regarding poverty measures. In practice equivalence scales are popular. Their implications are derived in

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[16] This condition is also proved in Hammond's (1977) Proposition 3.2 for social welfare functions.

Proposition 2

Let $D = \mathbb{R}_{++}$ *and let P be a smooth poverty measure,* **Z** *a vector of poverty lines,* **E** *an equivalent income function such that* $E_i(X_i) = X_i/m_i$ *for* $i = 2,...,N$ *, and* $m_1 = 1$*. Then the following statements are equivalent:*

(a) *P satisfies BTRT(***E***,***Z***).*

(b) $p^i(X_i, Z_i) = m_i\, p^1(X_i/m_i, Z_1)$ *for all* $X_i \in D$ *and* $Z_i = m_i Z_1$ *for* $i = 2,...,N$.

When (relative) equivalence scales are used we get a particular relationship[17] between the deprivation functions of a type *i*- household and of a single adult. Household *i*'s deprivation can be expressed directly by means of p^1 and the type-specific weights m_i. The relationship lends itself to a simple interpretation: If equivalence scales are employed, household *i* is equivalent to m_i (single) adults. Therefore household *i*'s deprivation also corresponds to the m_i-fold deprivation of a single adult having the equivalent income X_i/m_i and facing the poverty line Z_1. Then a single adult represents the basic unit of analysis and the poverty measure *P* can be expressed by the total deprivation of a homogeneous population consisting of equivalent adults. In this case it suggests itself to normalize the measure by the total sum of equivalent adults: $\Sigma_{i=1}^{N} m_i$.

For non-smooth measures things are a bit more complicated:

Proposition 2a

Let $D = \mathbb{R}_{++}$ *and let P be a regular poverty measure,* **Z** *a vector of poverty lines,* **E** *an equivalent income function such that* $E_i(X_i) = X_i/m_i$ *for* $i = 2,...,N$ *, and* $m_1 = 1$*. Define k by* $E_k(Z_k) := \max_{j=1,\dots,N} \{E_j(Z_j)\}$*. Then the following statements are equivalent:*

(a) *P satisfies BTRT(***E***,***Z***).*

(b) *There are constants* $\alpha_1,\dots,\alpha_N \in \mathbb{R}_+$ *such that* $p^i(X_i, Z_i) = \frac{m_i}{m_k} p^k(X_k, Z_k) - \alpha_i$ *for all* $X_i, X_k \in D$ *with* $E_i(X_i) = E_k(X_k)$ *and* $X_i \le Z_i$ *for* $i = 1,\dots,N$ ..

In this case the result is weaker: household *i*'s deprivation is only an affine transformation of type *k*'s deprivation function where household (type) *k* has the maximum poverty line (meas-

[17] Cf. Ebert (1997) in which separable welfare functions are considered.

ured by equivalent income). Here, of course, the living standards corresponding to the poverty lines are not necessarily identical (cf. also Proposition 1).

(b) **Using weights and equivalent incomes**

Next we suppose that the deprivation functions p^i for $i \geq 2$ are related to p^1 by means of weights and the equivalent income function. We can establish

Proposition 3

Let $D = \mathbb{R}_{++}$ *and let P be a smooth poverty measure,* **Z** *a vector of poverty lines, and* **E** *an equivalent income function and assume that there are weights* $w_2, \ldots, w_N \in \mathbb{R}_{++}$ *such that* $p^i\left(X_i, Z_i\right) = w_i\, p^1\left(E_i\left(X_i\right), Z_1\right)$ *for all* $X_i \in D$ *and* $i = 2,...,N$. *Then the following statements are equivalent:*

(a) *P satisfies BTRT(***E***,***Z***).*

(b) $E_i\left(X_i\right) = X_i / m_i$ *for all* $X_i \leq Z_i$, $Z_i = m_i Z_1$ *and* $m_i = w_i$ *for* $i = 2,...,N$.

For a smooth poverty measure it turns out that we have to employ relative equivalence scales and that the scale values have to be identical with the respective weights. Thus the use of weights a priori restricts the form of the equivalent income function for which the poverty measure can satisfy the BTRT(**E**,**Z**) principle and determines the scale values. The weight w_i implies that household *i* is represented by and has the needs of w_i adults. Then the household’s living standard can be represented by means of the equivalent scale $m_i = w$. This outcome is based on property (iv) imposed on equivalent income functions: It requires that arbitrary living standards (of different household types) can be compared. The property seems to be indispensable and can only be satisfied by relative equivalence scales.

Comparing Propositions 2 and 3 we recognize that equivalence scales call for weights and conversely. At first sight Proposition 3 resembles Proposition 3.3 in Hammond (1977), but the settings are different: In Hammond a weighted utilitarian welfare function is considered and living standard is represented by the utility functions involved. In our case we obtain a weighted sum of deprivation functions for the poverty measure. Living standards are described by an equivalent income function.

(c) **Invariance property**

Here we want to examine relative (absolute) measures and get:

Proposition 4

Let $D = \mathbb{R}_{++}$ $\left[D = \mathbb{R}\right]$ *and let P be a smooth relative* [*absolute*] *poverty measure (i.e., assume that there are functions* $g^1,...,g^N$ *such that* $p^i\left(X_i, Z_i\right) = g^i\left(X_i/Z_i\right)$ $\left[= g^i\left(Z_i - X_i\right)\right]$ *for all* $X_i \in D$ *and* $i = 1,...,N$ *),* **Z** *a vector of poverty lines, and* **E** *an equivalent income function. Then the following statements are equivalent:*

(a) *P satisfies BTRT(***E***,***Z***).*

(b) *There are constants* $m_2,\ldots,m_N$ $\left[a_2,\ldots,a_N\right]$ *such that* $E_i\left(X_i\right) = X_i/m_i$ $\left[= X_i - a_i\right]$ *for* $X_i \le Z_i$, $g^i\left(t\right) = m_i g^1\left(t\right)$ $\left[= g^1\left(t\right)\right]$ *for* $t \in \mathbb{R}_{++}$ $\left[t \in \mathbb{R}\right]$ *and* $Z_i = m_i Z_1$ $\left[Z_i = Z_1 + a_i\right]$ *for* $i = 2,\ldots,N$.

If *P* has the particular form of a relative[18] or absolute poverty measure the BTRT principle requires that the equivalent income function can be expressed by relative and, respectively, absolute equivalence scales. This outcome is a direct consequence of the invariance property for the equivalent income function: For a relative measure it has to be proportional, for an absolute one translatable. Only these specific forms of the equivalent income function are feasible.

In this case the deprivation functions can be described by

$$p^i\left(X_i, Z_i\right) = m_i p^1\left(X_i/m_i, Z_i/m_i\right) = m_i g^1\left(X_i/Z_i\right) = m_i g^1\left(\left(X_i/m_i\right)/Z_1\right)$$

and, respectively, by

$$p^i\left(X_i, Z_i\right) = p^1\left(X_i - a_i, Z_i - a_i\right) = g^1\left(Z_i - X_i\right) = g^1\left(\left(Z_1 + a_i\right) - X_i\right) \text{ for } i = 2,\ldots,N .$$

For a relative poverty measure household *i* is transformed into m_i equivalent adults. For an absolute one differences in needs are represented by a fixed amount of income (the absolute equivalence scale a_i). But in contrast to the situation discussed in Proposition 3 where the weights are given, the researcher is here allowed to choose the equivalence scale values as she likes.

4.2 Discussion

In subsection 4.1 we presented the characterization of several classes of equity-regarding poverty measures: the deprivation functions and the equivalent income function always have

[18] Relative poverty measures are also considered in a different framework in Ebert (2004).

to be consistent. Moreover, for smooth measures the poverty lines have to correspond to the same living standard. The details of the analysis, however, depend on the particular scenario considered. In the following we want to reconsider these results and their consequences in a more general setting.

(a) Equivalent income function

In view of the literature in which (relative) equivalence scale predominate it is worth demonstrating by an example[19] that the between-type regressive transfer principle can also be satisfied for (more) general forms of equivalent income functions, i.e., Proposition 1 holds generally.

Choosing $D = \mathbb{R}_{++}$ we define a poverty measure $\overline{P}$ by introducing

$$\overline{p}^i\left(X_i, Z_i\right) := \begin{cases} b_i \ln \dfrac{1-e^{-Z_i}}{1-e^{-X_i}} & \text{for } X_i \le Z_i \\ 0 & \text{for } X_i > Z_i \end{cases}$$

and an equivalent income function $\overline{\mathbf{E}}$ by

$$\overline{E}_i\left(X_i\right) := \ln\left(1+\left(e^{X_i}-1\right)/b_i\right)$$

for $i = 1, \ldots, N$, where the type-specific parameters $b_i \in \mathbb{R}_{++}$ and $b_1 = 1$.[20] $\overline{P}$ is a regular poverty measure.

The equivalent income function $\overline{\mathbf{E}}$ seems to be complicated, but it can be interpreted easily: We obtain $d\overline{E}_i\left(X_i\right)/dX_i \to 1/b_i$ for $X_i \to 0$ and $d\overline{E}_i\left(X_i\right)/dX_i \to 1$ for $X_i \to \infty$. The marginal increase in the equivalent income of a type *i*-household is approximately $1/b_i$ for low incomes. Therefore b_i has the same impact as a relative equivalence scale. For high incomes the marginal increase in equivalent income is approximately equal to unity. Then the difference $X_i - \overline{E}_i\left(X_i\right)$ is approximately constant and can be interpreted as an absolute equivalence scale. Thus the equivalent income function represents a combination of relative and absolute equivalence scales: The parameter b_i could be set equal to a relative equivalent scale m_i or to the number of persons in the household.

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19 For details see Appendix 2.

20 See Ebert (2000) for a discussion of an equivalent income function more general than $\overline{\mathbf{E}}$.

The poverty measure $\overline{P}$ satisfies the between-type regressive transfer principle with respect to $\overline{\mathbf{E}}$ and any vector of poverty lines **Z**. Since the equivalent income function $\overline{\mathbf{E}}$ is nonlinear and BTRT($\overline{\mathbf{E}}, \mathbf{Z}$) is fulfilled, the example demonstrates that the transfer principle does *not* require the use of relative or absolute equivalence scales.

(b) Empirical example reconsidered

Next we discuss the empirical example presented in section 3 in the light of the theoretical results. It turns out that it illustrates the propositions presented. The measure P^* is smooth. Therefore, according to Proposition 1, the condition

$$p_X^i\left(X_i, Z_i\right) = 2w_i\left(\frac{Z_i - X_i}{Z_i}\right)\left(-\frac{1}{Z_i}\right) \overset{!}{=} 2\left(\frac{Z_1 - X_i/m_i}{Z_1}\right)\left(-\frac{1}{Z_1}\right) = p_X^1\left(E_i\left(X_i\right), Z_1\right)$$

has to be satisfied if the poverty measure P^* is to fulfill BTRT(**E**,**Z**). It directly implies that $Z_i = m_i Z_1$ and $w_i = m_i$, i.e., the poverty levels must correspond to the same living standard and the weight for type *i* has to be identical with the corresponding equivalent scale value. Since we have $w_1 = m_1$, the principle BTRT(**E**,**Z**) requires that method (iii) is applied for weighting the contributions of the different household types to overall poverty.

Taking into account the equivalence scales we learn from Proposition 2 that the deprivation functions have to be related: $p^i\left(X_i, Z_i\right) = m_i p^1\left(X_i/m_i, Z_1\right)$, i.e., Proposition 2 also proves that method (iii) has to be employed if the poverty measure is to be equity-regarding. Indeed, the implications of the specific transfers A and B examined in section 3 illustrate this requirement. If we read the definition (3) differently and postulate that weights are to be used $\left(p^i\left(X_i, Z_i\right) = w_i p^1\left(E_i\left(X_i\right), Z_1\right)\right)$, the equivalent income function must be based on equivalence scales and the scale values have to be identical to the corresponding weights (Proposition 3). Finally, the measure P^* described in (3) is a relative measure. This invariance property can only be satisfied if equivalence scales are employed and method (iii) is used (Proposition 4).

(c) Equivalence scales

We consider the particular case that living standard is described by means of relative scales $m_1 = 1, \quad m_2, \ldots, m_N \in \mathbb{R}_{++}$. Then a smooth equity-regarding poverty measure is already completely determined by $p^1\left(X_1, Z_1\right)$ and the poverty line Z_1. For a heterogeneous population which has many households of the same type and for $Z_i = m_i Z_1 \ (i = 1, \ldots, N)$, the poverty measure can be written as[21]

$$P^*(\mathbf{X}, \mathbf{Z}) = \left(1/\Sigma\, n_i m_i\right) \sum_{i=1}^{N} \sum_{j=1}^{n_i} m_i\, p^1\left(X_j^i / m_i, Z_1\right). \qquad (6)$$

The particular weighting system (method (iii)) is an immediate consequence of the formulation of the BTRT principle[22] and of condition (5): The redistribution of household income is based on a comparison of living standards which are measured by *equivalent* income. For an increase in equivalent income by one unit household *i* needs m_i units of household income, i.e. the m_i-fold income a single adult needs. Therefore household *i* is treated like m_i (equivalent) adults in this framework. This fact is also reflected by the normalization used in (6).

When using equivalence scales we get a conflict between the BTRT-principle and the principle of individualism which postulates that the *individuals* form the basis of the analysis (cf. the discussion of this topic in Ebert (1997) and Shorrocks (2004)). Then all individuals count equally and the weights have to be equal to the number of individuals concerned (method (ii)). But if the weights are chosen according to method (ii) the BTRT principle can be violated by particular transfers (cf. Table 2): Weighting by the number of individuals who belong to the household is (always) compatible with the BTRT principle only if living standard can be represented by the household's *average* income – a situation which does not occur if there are increasing returns of household size. Thus in general one has to discard either the BTRT-principle or the principle of individualism.[23]

(d) Construction of measures

Finally it is interesting to discuss the construction of smooth[24] equity-regarding poverty measures. Proposition 1 provides some advice: We can choose an arbitrary equivalent income function **E** and the poverty line Z_1 which represents the living standard separating poor and nonpoor households. Furthermore the deprivation function for one household type can be determined arbitrarily. If e.g. $p^1(X_1, Z_1)$ is given we obtain the remaining deprivation func-

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21 For absolute measure the scales are given by $a_1 = 0, a_2, \ldots, a_N \in \mathbb{R}$ and we get an analogue to (6): $P(\mathbf{X}, \mathbf{Z}) = \Sigma_i \Sigma_j\, p^1\left(X_j^i - a_i, Z_1\right)$.

22 Cf. e.g. Pyatt (1990), Ebert (1997, 2004) and Ebert and Moyes (2003).

23 Shorrocks (2004) demonstrates in his investigation of welfare and inequality that both principles can be compatible if the assumptions of differentiability and separability are dropped.

24 If the measure is only regular, we can choose an arbitrary vector of poverty lines, a vector of constants $\alpha_1, \ldots, \alpha_N$, and p^k (instead of p^1) if *k* is defined by $E_k(Z_k) = \max\{E_i(Z_i)\}$.

tions from (5) – by using the equivalent income function **E**: In this case we have to integrate p_X^i (Proposition 1). For equivalence scales, i.e., if we a priori take $E_i\left(X_i\right) = X_i / m_i$, the deprivation functions can be defined directly by $p^i\left(X_i, Z_i\right) = m_i p^1\left(X_i / m_i, Z_1\right)$ (Proposition 2), see also (c) above. In order to introduce weights we have to proceed in the same way, and we have to employ the correct equivalence scales ($m_i = w_i$; Proposition 3). For relative (absolute) poverty measures we must use equivalence scales, but are free to choose the scale values (Proposition 4).

5. Conclusion

The paper has investigated the meaning of the BTRT principle for the measurement of poverty. Given an equivalent income function **E** the satisfaction of BTRT(**E**,**Z**) requires and is equivalent to some conditions on (the relation between) the poverty measure, equivalent income function, and poverty lines. In particular it turns out that for smooth measures the poverty lines of different types have to represent the same living standard. In principle one can always find an equity-regarding poverty measure for any concept of living standard.

There are several situations in which equivalence scales have to be employed when the BTRT principle is imposed. First, when weights are used in the definition of the deprivation functions, the equivalent income functions have to be based on equivalence scales. Second, it is often attractive to choose relative (absolute) poverty measures. In this case the BTRT principle can only be satisfied if living standard is expressed by means of relative (absolute) equivalence scales.

The choice of weights is a normative issue and may be debatable. But there are further reasons supporting the structure given in (6): Ebert (2005) demonstrates that the paradox of targeting – an increase in the needs of some group leads to a reduction of resources allocated to it in an optimal allocation (cf. Keen (1992)) – can only be avoided if the weights and equivalence scales are identical. Then an optimal program for the alleviation of poverty, which is based on uniform poll subsidies or guarantees minimum incomes, turns out to be horizontally equitable.

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Appendix 1: Proof of Propositions

Proof of Proposition 1

(a) $\Rightarrow$ (b):

We have to employ the fact that *P* satisfies BTRT(**E**,**Z**): consider **X** and **Y** such that

$$Y_i = X_i - \varepsilon, \quad Y_j = X_j + \varepsilon \text{ for } \varepsilon > 0$$

and $Y_k = X_k$ otherwise, and

$$E_i(Y_i) < E_i(X_i) \le E_j(X_j) < E_j(Y_j) \le \min\{E_i(Z_i), E_j(Z_j)\}.$$

Then $P(\mathbf{X},\mathbf{Z}) \le P(\mathbf{Y},\mathbf{Z})$.

This condition is equivalent to

$$\frac{p^i(X_i, Z_i) - p^i(X_i - \varepsilon, Z_i)}{\varepsilon} \le \frac{p^1(X_j + \varepsilon, Z_j) - p^1(X_j, Z_j)}{\varepsilon}.$$

For $\varepsilon \to 0$ we obtain

$$p_X^i(X_i, Z_i) \le p_X^j(X_j, Z_j) \text{ for } E_i(X_i) \le E_j(X_j) < \min\{E_i(Z_i), E_j(Z_j)\}.$$

Analogously we can prove that

$$p_X^j(X_j, Z_j) \le p_X^i(X_i, Z_i) \text{ for } E_j(X_j) \le E_i(X_i) < \min\{E_i(Z_i), E_j(Z_j)\}.$$

Continuity implies that

$$p_X^i(X_i, Z_i) = p_X^j(X_j, Z_j) \text{ for } E_i(X_i) = E_j(X_j) \le \min\{E_i(Z_i), E_j(Z_j)\}.$$

Now assume that *P* is a smooth measure and that $E_i(Z_i) < E_j(Z_j)$. When X_i tends to Z_i we obtain

$$E_i(X_i) \to E_i(Z_i) < E_j(Z_j) \text{ and } 0 = p_X^i(Z_i, Z_i) = p_X^j\left(E_j^{-1}(E_i(Z_i)), Z_j\right)$$

since p_X^i is continuous at $X_i = Z_i$.

By assumption we have $p_X^j\left(E_j^{-1}\left(E_i\left(Z_i\right)\right),Z_j\right)<0$ for $E_i\left(Z_i\right)<E_j\left(Z_j\right)$, i.e. we get a contradiction and $E_i\left(Z_i\right)\geq E_j\left(Z_j\right)$. Analogously we can show that $E_i\left(Z_i\right)\leq E_j\left(Z_j\right)$. Thus in particular $E_i\left(Z_i\right)=Z_1$.

(b) $\Rightarrow$ (a):

Use (b) and observe that $p_{XX}^i\geq 0$. □

Proof of Proposition 2 and Proposition 2a

(a) $\Rightarrow$ (b):

According to Proposition 1 BTRT(**E**,**Z**) implies that

$$p_X^i\left(X_i,Z_i\right)=p_X^j\left(\left(m_j/m_i\right)X_i,Z_j\right)$$

since $\left(E_i\left(X_i\right)=E_j\left(X_j\right)\Leftrightarrow X_j=\left(m_j/m_i\right)X_i\right)$.

Integration yields

$$\int_a^b p_X^i\left(X_i,Z_i\right)dX_i=\frac{m_i}{m_j}p^j\left(\frac{m_j}{m_i}X_i,Z_j\right)\Bigg|_{a.}^{b}$$

Now choose $a:=Z_i$ and $b:=X_i$ and assume that $Z_i/m_i\leq Z_j/m_j$. Then

$$p^i\left(X_i,Z_i\right)-p^i\left(Z_i,Z_i\right)=\frac{m_i}{m_j}p^j\left(\frac{m_j}{m_i}X_i,Z_j\right)-\frac{m_i}{m_j}p^j\left(\frac{m_j}{m_i}Z_i,Z_j\right)$$

which proves the claim since $p^i\left(Z_i,Z_i\right)=0$.

If P is smooth, the second term of the RHS vanishes.

(b) $\Rightarrow$ (a): Direct proof □

Proof of Proposition 3

(a) $\Rightarrow$ (b):

$$p^i\left(X_i,Z_i\right)=w_i\,p^1\left(E_i\left(X_i\right),Z_1\right)$$

implies that

$$p_X^i\left(X_i,Z_i\right)=w_i\,E_i'\left(X_i\right)p_X^1\left(E_i\left(X_i\right),Z_1\right).$$

Using Proposition 1(b) we get $w_i E_i'(X_i) = 1$ for $X_i \leq Z_i$. Thus there is $a_i \in \mathbb{R}$ s.t. $E_i(X_i) = X_i / w_i + a_i$.

If $D = \mathbb{R}_{++}$ we obtain $a_i = 0$ since $E_i(X_i) \to 0$ for $X_i \to 0$.

(b) $\Rightarrow$ (a): Direct proof □

Proof of Proposition 4

Suppose that P is a relative measure and satisfies BTRT(**E**,**Z**). Proposition 1 implies that $E_i(Z_i) = Z_1$ and that $E_i(\lambda Z_i) = \lambda Z_1$ for $\lambda > 0$ since P is a relative measure.

Then $E_i(\lambda Z_i) = \lambda Z_1 = \lambda E_i(Z_i)$ and E_i has to be proportional. Now apply Proposition 2.

Analogously, if P is an absolute measure. Proposition 1 implies that $E_i(Z_i) = Z_1$ and $E_i(Z_i + \alpha) = Z_1 + \alpha$.

Thus

$$E_i(Z_i + \alpha) = Z_1 + \alpha = E_i(Z_i) + \alpha \text{ and } E_i'(Z_i + \alpha) = 1.$$

Then E_i has to be linear and there is $a_i \in \mathbb{R}$ s.t. $E_i(Z_i) = Z_i - a_i$. The rest follows from (5) by integration. □

Appendix 2: Example 1

(i) The equivalent income function

$$E_i(X_i) = \ln\left(1 + \left(e^{X_i} - 1\right)/b_i\right)$$

$$X_i \to 0 \quad \Rightarrow \quad E_i(X_i) \to 0$$

$$\frac{dE_i(X_i)}{dX_i} = \frac{1}{1 + \left(e^{X_i} - 1\right)/b_i} \frac{e^{X_i}}{b_i} = \frac{1}{b_i} \frac{e^{X_i}}{1 + \left(e^{X_i} - 1\right)/b_i}$$

$$X_i \to 0 \quad \Rightarrow \quad dE_i(X_i)/dX_i \to 1/b_i$$

$$X_i \to \infty \quad \Rightarrow \quad dE_i(X_i)/dX_i \to 1$$

(ii) Deprivation functions

$$p^i(X_i, Z_i) = b_i \ln\frac{1 - e^{-Z_i}}{1 - e^{-X_i}} = b_i \ln\left(1 - e^{-Z_i}\right) - b_i \ln\left(1 - e^{-X_i}\right)$$

$$p_X^i(X_i, Z_i) = -b_i \frac{1}{1 - e^{-X_i}}\left(-e^{-X_i}\right)(-1)$$

$$= -b_i \frac{e^{-X_i}}{1 - e^{-X_i}} = -b_i \frac{1}{e^{X_i} - 1} < 0$$

$$p_{XX}^i(X_i, Z_i) = -b_i \frac{(-1)}{\left(e^{X_i} - 1\right)^2} e^{X_i} = b_i \frac{e^{X_i}}{\left(e^{X_i} - 1\right)^2} > 0$$

(iii) BTRT(**E**,**Z**)

We consider

$$p_X^1\left(E_i(X_i), E_i(Z_i)\right) = -b_1 \frac{1}{e^{E_i(X_i)} - 1}.$$

Now

$$e^{E_i(X_i)} = e^{\ln\left(1 + \left(e^{X_i} - 1\right)/b_i\right)} = 1 + \left(e^{X_i} - 1\right)/b_i.$$

Therefore

$$p_X^1\left(E_i(X_i), E_i(Z_i)\right) = -b_1 \frac{1}{\left[1 + \left(e^{X_i} - 1\right)/b_i\right] - 1} = -b_i \frac{1}{e^{X_i} - 1} = p_X^i(X_i, Z_i)$$

since $b_1 = 1$.