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THE INEQUALITY PROCESS AS A WEALTH MAXIMIZING PROCESS

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Abstract. The One Parameter Inequality Process (OPIP) long predates the Saved Wealth Model (SWM) to which it is isomorphic up to a different choice of stochastic driver of wealth exchange. Both are stochastic interacting particle system intended to model wealth and income distribution. The OPIP and other versions of the Inequality Process explain many aspects of wealth and income distribution but have gone undiscussed in econophysics. The OPIP is a jump process with a discrete 0,1 uniform random variate driving the exchange of wealth between two particles, while the SWM, as an extension of the stochastic version of the ideal gas model, is driven by a continuous uniform random variate with support at [0.0, 1.0]. The OPIP's stationary distribution is a Lévy stable distribution attracted to the Pareto pdf near the (hot) upper bound of the OPIP's parameter, ω, and attracted to the normal (Gaussian) pdf toward the (cool) lower bound of ω. A gamma pdf model approximating the OPIP's stationary distribution is heuristically derived from the solution of the OPIP. The approximation works for $\omega < .5$, better as ω -> 0. The gamma pdf model has parameters in terms of ω. The Inequality Process with Distributed Omega (IPDO) is a generalization of the OPIP. In the IPDO each particle can have a unique value of its parameter, i.e., particle i has ω_i, The meta-model of the Inequality Process implies that smaller ω is associated with higher skill level among workers. This hypothesis is confirmed in a test of the IPDO. Particle wealth gain or loss in the OPIP and IPDO is more clearly asymmetric than in the SWM ($\lambda \neq 0$). Time-reversal asymmetry follows from asymmetry of gain and loss. While the IPDO scatters wealth, it also transfers wealth from particles with larger ω to those with smaller ω, particles that according to the IPDO's meta-model are more productive of wealth, nourishing wealth production. The smaller the harmonic mean of the ω_i's in the IPDO population of particles, the more wealth is concentrated in particles with smaller ω, the less noise and the more ω signal there is in particle wealth, and the deeper the time horizon of the process. The IPDO wealth concentration mechanism is simpler than Maxwell's Demon.

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Keywords: Competition, Gamma pdf, Income distribution, Robust loser, Techno-cultural evolution, Wealth maximization

1 Introduction[1]

The simplest version of the Inequality Process, the One Parameter Inequality Process (OPIP) [1-23] has different properties from the Saved Wealth Model (SWM) of Chakraborti and Chakrabarti [24], examined and elaborated in[24-30] although the two are isomorphic up to their stochastic drivers and Lux [1] judges the two models to be "essentially equivalent". The respective meta-models differ. The choice of the stochastic driver of wealth exchange in each follows from the meta-model of each. A meta-model is a set of understandings about empirical referents, variables, hypotheses, and tests. The present paper shows that the OPIP and a generalization of it, the Inequality Process with Distributed Omega (IPDO), form a theory of wage income distribution as a byproduct of wealth maximization while the SWM does not. Before Lux [1], the Inequality Process was undiscussed in the econophysics literature.

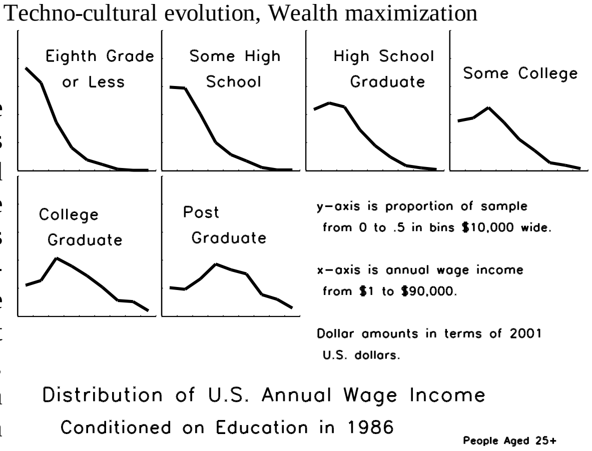

Figure 1
Source: Author's estimates from the March CPS. See Appendix A.

[1] I thank Thomas Lux (2005) [1] for informing the International Workshop on the Econophysics of Wealth Distributions, Kolkata, March 15-19, 2005, a symposium on the Saved Wealth Model (SWM) of Chrakraborti and Chakrabarti [22], of the similarity between it and the Inequality Process. Thanks to Kenneth Land, Duke University,and Thomas Lux, University of Kiel, for their comments on this paper. The author is responsible for any error. This paper is a revision of that presented to the meeting of the Society for Anthropological Science, February 2005, Santa Fe, New Mexico, entitled "Speculation: The Inequality Process is the Competition Process Driving Human Evolution".

2 Comparison of Saved Wealth Model (SWM) to the One Parameter Inequality Process (OPIP)

The OPIP and the Saved Wealth Model (SWM) are both stochastic interacting particle system models that scatter a positive quantity, 'wealth'. The particles represent people with differing amounts of wealth. The models share assumptions of the ideal gas model, i.e., an isolated population of particles, random pairing of particles for exchange of a positive quantity, the sum of which after the exchange equals the sum before. The OPIP is a jump process with a discrete 0,1 uniform random variate driving the exchange of wealth between particles, while the SWM as an extension of the ideal gas model [31], uses a continuous uniform random variate with support at [0.0, 1.0] to drive wealth exchange. The SWM subsumes the ideal gas model as a special case. The difference between the continuous uniform random variate with support at [0.0, 1.0] of the SWM and the discrete 0,1 uniform random variate of the OPIP may appear inconsequential but that appearance is misleading. The OPIP's discrete stochastic driver highlights the relationship of its parameter to statistics of income, particularly those of wage income conditioned on education, whereas the continuous random variate of the SWM obscures that relationship.

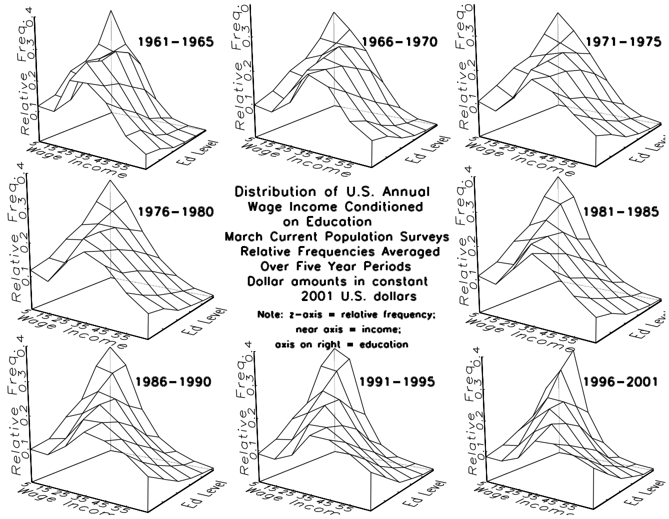

Figure 2
Source: Author's estimates from the March CPS. See Appendix A.

The Inequality Process is abstracted from the Surplus Theory of Social Stratification in economic anthropology [3, 32]. The Surplus Theory explains why hunter-gatherer society, viewed in anthropology as the most egalitarian societal form, turned into the chiefdom, the society of the god-king, viewed in anthropology as the most inegalitarian societal form. This striking transformation occurred whenever hunter/gatherer populations acquired a food surplus, usually through the acquisition of agriculture. This transformation occurred in populations far removed from each other in time, place, culture, and race, i.e., this transformation is one of the few universals of social science. The Surplus Theory accounts for this transformation as the result of greed, chance, and competition. However, while the Surplus Theory offers a parsimonious explanation of the transformation, it fails to explain why inequality in the sense of concentration, as measured for example by the Gini concentration ratio of wealth, decreases at higher techno-cultural stages than the chiefdom, particularly post Industrial Revolution [33]. Lenski [32] advances an explanation of decreasing inequality with techno-cultural evolution beyond the chiefdom: more skilled workers are able to keep a larger share of the wealth they produce. The Inequality Process models Lenski's speculative extension of the Surplus Theory.

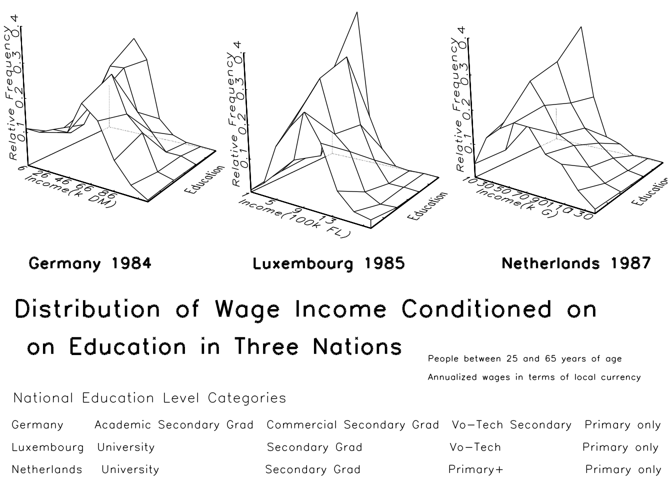

Figure 3
Source: Author's estimates from surveys of the Luxembourg Income Study. See [35].

The OPIP's 0,1 discrete uniform stochastic driver of wealth transfer in an encounter between two particles follows from the underlying verbal theory of a competition in which chance determines whether a win or a loss occurs but once a loss occurs its consequence is pre-determined. Losers lose a fixed proportion of their wealth, ω. A winner gains a random amount of wealth. Winning and losing in the OPIP are asymmetric. There is no asymmetry in particle collisions in the ideal gas model on which the SWM is based. Since the OPIP's meta-model says that more skilled workers lose less wealth in competition for wealth, (1-ω) represents worker skill level. The meta-model of the OPIP requires it to explain the course of the Gini concentration ratio of personal wealth over techno-cultural evolution, and requires a generalization of the OPIP that allows particles to have unique values of ω, i.e., particle i with ω_i, to replicate features of the distribution of wage income conditioned on education seen in figures 1, 2, and 3. Figure 1 shows the relationship between education and the distribution of wage income conditioned on education in the U.S. in 1985 as estimated from March 1986 Current Population Survey (CPS) data. See [34] and Appendix A. Figure 2 demonstrates the stability of this relationship over four decades in the U.S. Figure 3 suggests that the relationship between education and wage income distribution is not unique to the U.S. [35].

The transition equations of the Saved Wealth Model (SWM) [24] for the exchange of wealth between a pair of particles in the notation of [27] are:

$$
\begin{aligned}
x'_i &= \lambda x_i + \epsilon (1 - \lambda)(x_i + x_j) \\
x'_j &= \lambda x_j + (1 - \epsilon)(1 - \lambda)(x_i + x_j)
\end{aligned}
\qquad (1a,b)
$$

where x'_i is particle i's wealth after an encounter with particle j, x_i particle i's wealth before the encounter and ϵ is a continuous, uniform i.i.d random variate with support at [0.0,1.0]. λ, called "savings", is the parameter of the SWM. λ is the fraction of each particle's wealth not available for transfer in an encounter with another particle:

$$0.0 \leq \lambda < 1.0$$

Particles are randomly selected for pairwise interaction.

Making the substitutions: $\lambda \rightarrow (1 - \omega)$ and $\epsilon \rightarrow d_t$ where:

$$0.0 < \omega < 1.0$$

and:

$$d_t = \begin{cases} 1 & \text{with probability .5 at time t} \\ 0 & \text{otherwise} \end{cases}$$

yields the one parameter Inequality Process (OPIP) [7, 13, 16]:

$$
\begin{aligned}
x_{it} &= (1 - \omega)\, x_{i(t-1)} + d_t\, \omega \left(x_{i(t-1)} + x_{j(t-1)}\right) \\
x_{jt} &= (1 - \omega)\, x_{j(t-1)} + (1 - d_t)\, \omega \left(x_{i(t-1)} + x_{j(t-1)}\right)
\end{aligned}
$$

(2a,b)

(2a,b) differs from (1a,b). While ϵ is a continuous uniform random variate with support at [0.0,1.0], d_t is a discrete uniform random variate taking on the values 0 or 1, a Bernoulli variable. The difference between the intervals on which λ and ω are defined follows from the respective choice of stochastic driver. The OPIP's meta-model implies that (1-ω) = 0, the image of λ = 0 is meaningless. Its image, λ = 0, in the SWM is, however, well defined, where λ = 0 is equivalent to the stochastic version of the ideal gas model [31]. The assumption of an even number of particles in the OPIP population and the simultaneity of particle encounters in the OPIP are simplifications that permit an approximate solution for the wealth of particle i at time t, x_{it}.

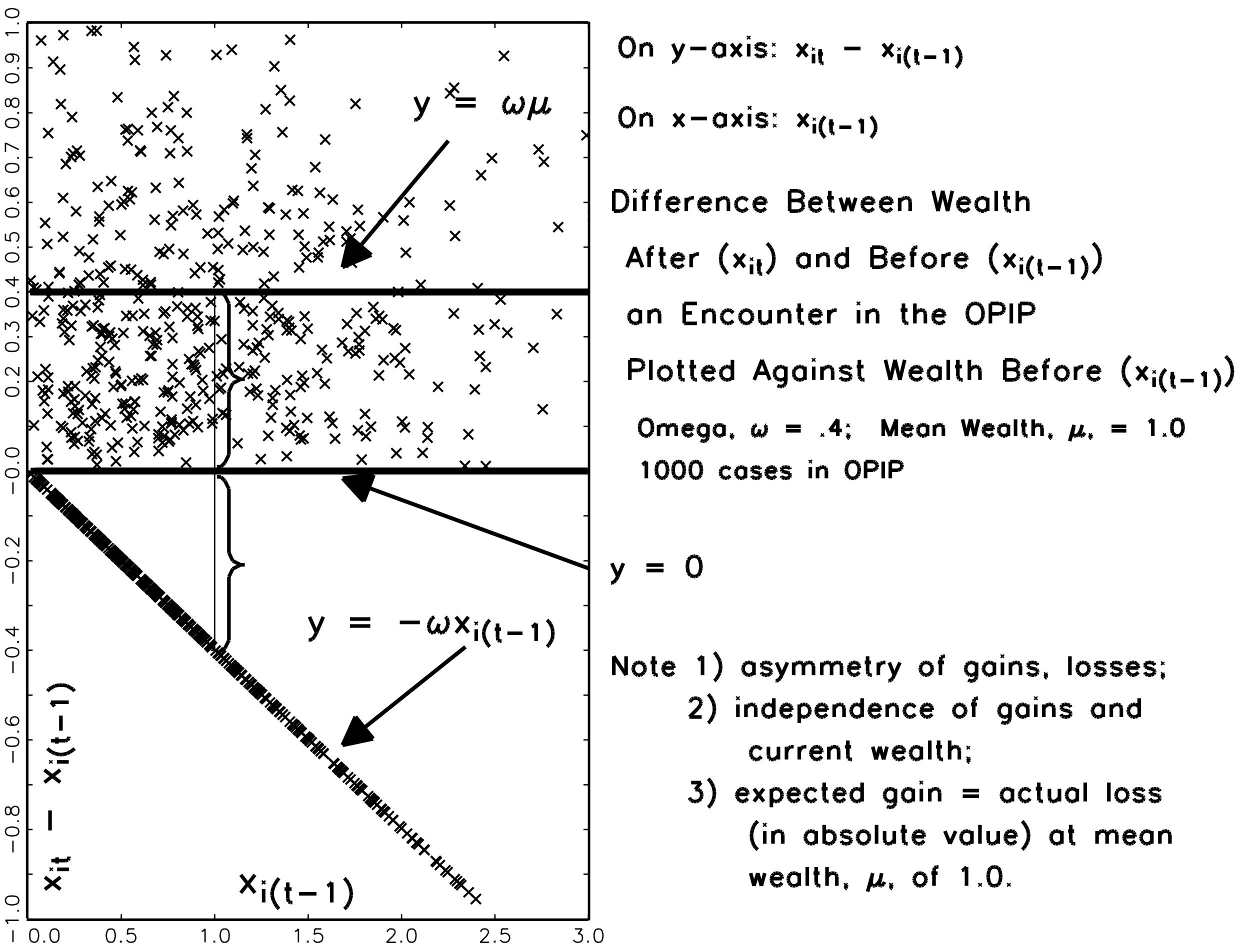

Figure 4: Scattergram of forward differences of wealth in OPIP against wealth

Since the SWM is an extension of the ideal gas model, the SWM's concept of temperature is analogous to that of the ideal gas model: $(1-\lambda)\,\mu$, where μ is mean wealth. $(1-\lambda)\,\mu$ is the mean of wealth exchanged between all particles if all particles were paired for simultaneous encounters. The OPIP's image of the SWM's $(1-\lambda)\,\mu$ is $\omega\mu$. However, the properties of the OPIP are unaffected by μ, which the OPIP takes as an exogenous variable. The OPIP is μ-symmetric. ω is the analogue of temperature in the OPIP since the properties of $\omega\mu$ in the OPIP depend on ω. Note that the analogue of temperature in the OPIP is bounded from above.

In the OPIP, particle i wins if $d_t = 1$ with probability 1/2 and loses otherwise. There is no outcome in-between a win or a loss. The two outcomes are asymmetric in the OPIP, as you can see in figure 4. In the event that particle i wins, it gains an ω share of its competitor's wealth. The gain for particle i is a random variable whose expectation is $\omega\mu$. On the other hand, if particle i loses, it loses a fixed share of its wealth, an ω share of its wealth, which from the point of view of particle i is a determinate outcome. The clarity of the asymmetry between winning and losing in the OPIP does not depend on the magnitude of ω. This asymmetry of the OPIP would be obscured if d_t were replaced by the SWM random variate, ϵ. In the SWM the asymmetry of winning and losing must be estimated through ϵ's noise and so does depend on the magnitude of λ.

The asymmetry of winning and losing in the OPIP provides time reversal asymmetry for particle wealth. In this respect the Inequality Process differs from the ideal gas model (the SWM for $\lambda = 0$). The OPIP also differs from the SWM for $\lambda > 0$ to the extent that ϵ obscures the difference between the λ and 0. The direction of time in the OPIP is readily ascertained by a glance at the time-series of an OPIP particle's wealth. The arrow of time points toward wealth amounts that are a constant fraction of a time-adjacent larger wealth amount. ω can be calculated from this time-series if ω is unknown. The corresponding determination in the SWM can only be made statistically, i.e., with information on a sample of particle wealth time-series, and with increasing difficulty as λ becomes small.

The generalization of the OPIP in which particles may have distinct values of ω, each particle i with an ω_i, is called the Inequality Process with Distributed Omega (IPDO). In the IPDO, ω_i can be calculated from the time-series of wealth amounts held by particle i. The direction of time can be ascertained from this time-series as well, if unknown. Angle [13, 16] has shown that ω_i can be estimated despite timewise aggregation of observations on particle i's time-series of wealth amounts.

3 The Stationary Distribution of the OPIP is Not a Gamma PDF[2]

The gamma pdf can be obtained by maximizing the entropy of the distribution of wealth of the particles in the population of a particle system subject to certain equality constraints. It is shown in this section of the paper that the required equality constraints cannot be derived from the OPIP. The OPIP scatters wealth since the absolute value of the expected difference of wealth between two particles after an encounter is less than before. The difference between particles x_{it} and x_{jt} in the OPIP is from (2a,b):

$$\begin{aligned}(x_{it} - x_{jt}) \;=\; & (x_{i(t-1)} - x_{j(t-1)}) \\ & + \; 2d_t\,\omega\, x_{j(t-1)} \\ & - \; 2(1-d_t)\,\omega\, x_{i(t-1)}\end{aligned} \tag{3}$$

Assuming that $x_{i(t-1)}$ and $x_{j(t-1)}$ are known, since given longitudinal survey data on income and wealth, data on these variables may exist, and given that $E[d_t] = ½$:[3]

$$E[x_{it} - x_{jt}] \;=\; (1-\omega)(x_{i(t-1)} - x_{j(t-1)}) \tag{4}$$

The expected value of the difference in wealth between two particles after their encounter has the same sign as the difference before the encounter diminished in absolute value by an ω proportion.

The distribution of x, wealth, in the OPIP can be approximated by a gamma probability density function (pdf), but the distribution is not gamma.[4] The gamma pdf is defined by:

$$f(x) \;\equiv\; \frac{\lambda^{\alpha}}{\Gamma(\alpha)} x^{\alpha-1} e^{-\lambda x} \tag{5}$$

where:

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[2] Based on Angle [5].

[3] '$E[d_t]$' is used to express the mathematical expectation of d_t.

[4] Dagum (1977) [36] cites March (1898) [37] as the first published instance of the gamma pdf being used to model a wage distribution. Such applications of the gamma pdf since have been desultory, most appearing in the 1970's, e.g., Peterson and von Foerster (1971) [38], Salem and Mount (1974) [39], Shorrocks (1975) [40], and Boisvert (1977) [41]. McDonald and Jensen (1979) [42], assuming the relevance of a two parameter gamma pdf model to empirical income distribution, derive expressions for the statistics of inequality of a gamma pdf in terms of its parameters. Cowell (1977) [43] mentions the gamma pdf as a model of income distribution, if not the best known. Kleiber and Kotz [23] discuss the gamma pdf and related pdf's as a parametric model for income distribution, but still following, as in Cowell's discussion, the Pareto and lognormal pdfs, which have figured more prominently in econometric practice.

$$\begin{aligned} x &> 0 \\ \alpha &> 0 \\ \lambda &> 0 \\ \alpha &\equiv \text{the shape parameter} \\ \lambda &\equiv \text{the scale parameter} \end{aligned}$$

Figure 5 shows the effect of α on the shape of the function for λ fixed at 1.0. The use of the Greek letter, λ, to denote the gamma's scale parameter is conventional and not related to the use of the letter in the SWM. A comparison of figures 1, 2, and 3 with 5 suggests a relationship between the education level of a worker and the shape parameter, the α, of a gamma pdf fitted to wage income conditioned on education.

The entropy statistic, H:

$$H \equiv -\sum_{i=1}^{I} p_i \ln(p_i) \tag{6}$$

is defined here in terms of relative frequency bins for the distribution of wealth in the OPIP. The argument of the entropy statistic is the relative frequency p_i of the wealth amount, x_i, where x_i is the mean of wealth in the i[th] relative frequency bin of fixed width. Given a finite population of particles of size N, the first constraint on the maximization of the entropy statistic to obtain a gamma pdf of the stationary distribution of wealth is that of the physical isolation of the particles and their persistence through time:

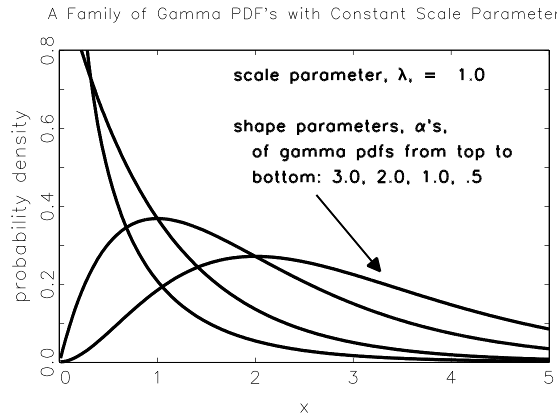

Figure 5: Gamma pdfs with common scale parameter, λ = 0, and different shape parameters

$$\sum_{i=1} \frac{n_i}{N} = \sum_{i=1}^{I} p_i = 1.0 \tag{7}$$

where n_i is the number of particles with the wealth that puts them into the i[th] relative frequency bin. The second constraint is the constancy of total wealth in the population of particles due to the fact that wealth is neither created nor destroyed in wealth exchanges in the Inequality Process, as can be seen by adding (2a) to (2b):

$$x_{it} + x_{jt} = x_{i(t-1)} + x_{j(t-1)} \tag{8}$$

The second constraint is:

$$\sum_{i=1} p_i x_i = \mu \tag{9}$$

where μ is the mean of wealth in the population of particles.

Maximizing the entropy statistic subject to these two equality constraints, which obtain in the ideal gas model, yields a negative exponential distribution of x. Maxentropically obtaining the gamma pdf instead requires a third constraint (Kapur, [44]):

$$\sum_{i=1} p_i \ln(x_i) = \tilde{\mu} \tag{10}$$

where $\tilde{\mu}$ is the mean of the natural logarithm of particle wealth. This constraint is exactly satisfied if:

$$\ln(x_{it}) + \ln(x_{jt}) = \ln(x_{i(t-1)}) + \ln(x_{j(t-1)}) \tag{11}$$

implying:

$$x_{it} x_{jt} = x_{i(t-1)} x_{j(t-1)} \tag{12}$$

in the interaction between a pair of particles in the OPIP, (2a,b). Taking earlier values, $x_{i(t-1)}$, as known, and later values, x_{it}, as unknown, the two constraints, equations #8 and #12, together imply:

$$x_{it} = \frac{-\left(x_{i(t-1)} + x_{j(t-1)}\right) \pm \left(x_{i(t-1)} - x_{j(t-1)}\right)}{-2} \tag{13}$$

i.e., either $x_{it} = x_{j(t-1)}$, which is not in general true, or $x_{it} = x_{i(t-1)}$, i.e., wealth amounts do not change, which is false. So the OPIP (2a,b) does not have a stationary distribution that is exactly gamma.

4 For Large ω the Stationary Distribution of the OPIP is Approximately Pareto[5]

Let particle i be the general particle in the population of the OPIP (2a,b). The derivation of approximations to the stationary distribution of this process follows from the solution of (2a,b) for particle i's current wealth at time t, x_{it}, in terms of the parameter, ω, the stochastic variables, the d_t's (Bernoulli variables), and the wealth of particles j, k, l, ... encountered by particle i at time t, t-1, t-2, ... This solution is found by back-substitution:

$$\begin{aligned} x_{it} = \; & x_{j(t-1)}\omega d_{it} \\ & + x_{k(t-2)}\omega d_{i(t-1)} \left[1-\omega(1-d_{it})\right] \\ & + x_{l(t-3)}\omega d_{i(t-2)}\left[1-\omega(1-d_{it})\right]\left[1-\omega(1-d_{i(t-1)})\right] \\ & + x_{m(t-4)}\omega d_{i(t-3)}\left[1-\omega(1-d_{it})\right]\left[1-\omega(1-d_{i(t-1)})\right]\left[1-\omega(1-d_{i(t-2)})\right] \\ & + x_{n(t-5)}\omega d_{i(t-4)}\left[1-\omega(1-d_{it})\right]\left[1-\omega(1-d_{i(t-1)})\right]\left[1-\omega(1-d_{i(t-2)})\right]\left[1-\omega(1-d_{i(t-3)})\right] \\ & + \dots \dots \dots \\ & + x_{\beta 1}\omega d_{i2}\left[1-\omega(1-d_{it})\right]\left[1-\omega(1-d_{i(t-1)})\right]\left[1-\omega(1-d_{i(t-2)})\right] \dots \dots \left[1-\omega(1-d_{i3})\right] \\ & + x_{\alpha 0}\omega d_{i1}\left[1-\omega(1-d_{it})\right]\left[1-\omega(1-d_{i(t-1)})\right]\left[1-\omega(1-d_{i(t-2)})\right] \dots \dots \left[1-\omega(1-d_{i2})\right] \\ & + x_{i0}\left[1-\omega(1-d_{it})\right]\left[1-\omega(1-d_{i(t-1)})\right]\left[1-\omega(1-d_{i(t-2)})\right] \dots \dots \left[1-\omega(1-d_{i1})\right] \end{aligned} \tag{14}$$

At time t, all stochastic events prior to d_t have been realized as 0's or 1's so the RHS of (14) equals:

$$\begin{aligned} x_{it} = \; & x_{j(t-1)}\omega d_{it} \\ & + x_{k(t-2)}\omega d_{i(t-1)} (1-\omega)^{1-d_{it}} \\ & + x_{l(t-3)}\omega d_{i(t-2)}(1-\omega)^{2-d_{it}-d_{i(t-1)}} \\ & + \dots \dots \dots \\ & + x_{\beta 1}\omega d_{i2}\,(1-\omega)^{(t-2)-\sum_{\tau=0}^{t-3} d_{i(t-\tau)}} \\ & + x_{\alpha 0}\omega d_{i1}\,(1-\omega)^{(t-1)-\sum_{\tau=0}^{t-2} d_{i(t-\tau)}} \\ & + x_{i0}\,(1-\omega)^{t-\sum_{\tau=0}^{t-1} d_{i(t-\tau)}} \end{aligned} \tag{15}$$

Particle i's wealth at time t, x_{it}, in (15) is a sum of ω sized "bites" taken by particle i out of the wealth of competitor particles at previous times multiplied by (1-ω) raised to the power of the sum of particle i's later losses, i.e., particle i's

[5] Based on Angle [6, 12, 15].

current wealth, x_{it}, is the sum of gains from past wins that particle i did not later lose. There is a run structure in (15) made clearer by re-writing (15) as:

$$
\begin{aligned}
x_{it} \;=\;& x_{j(t-1)}\omega d_{it} \\
&+\; x_{k(t-2)}\omega d_{i(t-1)}\;(1-\omega)^{(1,\ \text{if loss at t})} \\
&+\; x_{l(t-3)}\omega d_{i(t-2)}(1-\omega)^{(\sum \text{losses at t, t-1})} \\
&+\; \ldots\ \ldots\ \ldots \\
&+\; x_{\beta 1}\omega d_{i2}\;(1-\omega)^{(\sum \text{ losses time 3 until time t})} \\
&+\; x_{\alpha 0}\omega d_{i1}\;(1-\omega)^{(\sum \text{ losses time 2 until time t})} \\
&+\; x_{i0}\;(1-\omega)^{(\sum \text{ losses time 1 until time t})}
\end{aligned}
\qquad (16)
$$

When ω is large and (1-ω) small, particle i's wealth, x_{it}, in (16) can be approximated as the sum of winnings from competitors in a run of consecutive wins moving backward in time because the first loss encountered moving backward in time from the present, time t, stops the run by almost erasing particle i's wealth. Consequently:

$$x_{it} \approx \omega\left(x_{j(t-1)} + x_{k(t-2)} + x_{l(t-3)} + \ldots\ldots\right) \qquad (17)$$

x_{it} is approximately a Lévy stable variable where the number of summands on the RHS of (17), n, is distributed as a geometric probability function:

$$P(n = k) = (1 - p)\, p^{k} \qquad (18)$$

where p is the probability of a win and k is the number of wins. p in the OPIP equals ½. (17) is distributed, approximately, as a Pareto pdf [45]:

$$f(x) \equiv \alpha\, k^{\alpha} x^{-\alpha-1} \qquad (19)$$

where α is a parameter which just happens to be denoted by the same Greek letter as a parameter of the gamma pdf and k is the minimum of x:

$$
\begin{aligned}
\alpha &> 0 \\
k &> 0 \\
x &\geq k
\end{aligned}
$$

So the Pareto pdf is an attractor of the OPIP as ω → 1.0. This result should be compared with the stationary distribution of the SWM as (1-λ) → 1.0, the image in the SWM of the OPIP's ω → 1.0. The SWM's stationary distribution (1-λ) → 1.0 is a negative exponential pdf, a pdf with a lighter right tail than that of the Pareto. When ω is large and (1-ω) small, particle memory is short since the first loss encountered, moving back in time on the RHS of (16), erases nearly all the information accumulated in the prior history of losses encoded in a particle's wealth. The probability that a consecutive run of wins is no longer than τ, τ = 1,2, 3,....., is $1 - (½)^{\tau}$ which quickly approaches 1.0 given independent probabilities of winning equal to ½. When mean ω is large and mean (1-ω) is small in the IPDO, any difference between ω_i's will be difficult to distinguish using particle wealth, most of which is success runs produced by the Bernoulli variables, i.i.d. random variables, i.e., just noise.

5 For Small ω the Stationary Distribution of the OPIP is Approximately Gamma[6]

Conversely, when ω is small and (1- ω) large, a) losing an encounter does not by itself erase a particle's wealth and x_{it} can be approximated as the sum of winnings in a run of wins backward in time tolerating some intervening losses; b) the memory of the process is longer and there is more information about particle i's history in x_{it}; c) a particle's wealth is more due to small ω (wealth retention after a loss) than to consecutive wins in encounters, i.e., due to a determinate, unvarying characteristic of the particle rather than chance. As ω becomes small, the RHS of (15) loses its run-like character and becomes a sum of small random increments. Consequently, given the central limit theorem, as ω becomes small, the stationary distribution of x_{it} converges to a normal distribution.

When ω is is sufficiently small, x_{it} can be approximated by the RHS of (20):

$$x_{it} \approx \omega\mu \left[\begin{array}{c} d_t \\ + d_{(t-1)}(1-\omega)^{1-d_t} \\ + d_{(t-2)}(1-\omega)^{2-d_t-d_{(t-1)}} \\ + d_{(t-3)}(1-\omega)^{3-\sum_{m=0}^{2} d_{(t-m)}} \\ + \ldots\ldots\ldots \end{array} \right] \tag{20}$$

(20) is (15) with μ substituted for the wealth of particles that particle i encountered in the past. (20) can be demonstrated numerically to be an adequate approximation to (15) for many purposes when ω < .5.

The infinite series in the brackets on the RHS of (20) behaves much like the sum of a finite sequence of varying length of unweighted Bernoulli variables. The weight on each of the Bernoulli variables of (20) is (1-ω) raised to the power of losses occurring later in time. If there are no losses later in time, the weight is 1.0. The smaller (1-ω), the fewer losses need occur before any gain from a previous win has been erased. The weights allow particle i to keep gains as long as particle i wins, but given losses, a small number when (1-ω) is small, a larger number when (1-ω) is large, gains occurring before those losses are erased. (1-ω) determines a time horizon from the point of view of the present, time t, looking back into the past.

The infinite series of weighted Bernoulli variables in the brackets on the RHS of (20) can be approximated by summing a finite sequence of unweighted Bernoulli variables running from the present, t, back to t - τ in the past:

$$d_t + d_{t-1} + d_{(t-2)} + \ldots\ldots + d_{(t-\tau)} \tag{21}$$

Numerically, (21) becomes a better approximation to the series in the bracket on the RHS of (20) as ω decreases and τ increases (moving backward in time from the present) until particle i encounters enough losses to erase gains from encounters at earlier times. Let N be the minimum number of discrete losses that approximates the power of (1-ω) so that $(1-\omega)^N$ is negligibly different from zero. The number of wins of particle i, k, must be the same in (20) and (21).

[6] Based on Angle [6, 12, 15].

If N=1, one loss essentially wipes out any contribution from previous wins in the infinite series of (20), i.e., ω is so close to 1.0 that:

$$(1-\omega)^{(1-d_t)} \approx 0$$

when $d_t = 0$. N, like ω, is a parameter. N is the number of losses that makes prior wins irrelevant. N=1 corresponds to a (1-ω) close to 0.0 and the sum (21) is the number of consecutive wins moving backward in time from the present to time t-τ. Where N=1, a wealth of 0, i.e, k = 0, is approximated in (21) by a sequence of two Bernoulli variables, one at the present time, d_t, the other at the time previous, $d_{(t-1)}$. d_t must be a loss, else k > 0. $d_{(t-1)}$ is a loss too. It is the loss required by making N=1, since:

$$(1-\omega)^{(1-d_t)}$$

appears in the series in the bracket on the RHS of (20) at time (t-1) and could not approximate zero until time (t-1) because it appears as a cofactor of $d_{(t-1).}$ The effect of (1-ω) being small makes the question of whether $d_{(t-1)}$ equals 1 or 0 irrelevant in (20) since multiplication by small (1-ω) approximates a loss even it is a win. This effect must be represented in (21) by a loss: $d_{(t-1)} = 0$.

Where N = 1 and k = 1, (21) becomes the sum of three Bernoulli variables with realizations:

$$d_t = 1$$
$$d_{(t-1)} = 0$$
$$d_{(t-2)} = 0$$

The win at time t provides the k = 1 in (21) when this sequence of Bernoulli variables is summed in (21). The loss at t-1 must occur; else, because a win at time t-1 in (20) would be multiplied by:

$$(1-\omega)^{(1-d_t)} = 1.0$$

and k > 1, contrary to the hypothesis that k = 1. (1-ω) goes into effect at t-2 in (20), where because of a win at t and a loss at t-1:

$$(1-\omega)^{2-d_t-d_{t-1}} = (1-\omega)$$

In (20) it does not matter whether the Bernoulli variable at t-2, $d_{(t-2)}$ is a win or a loss, since multiplication of a win at t-2, $d_{(t-2)} = 1.0$, by (1-ω) approximates zero, a loss. This situation is modeled in (21) by $d_{(t-2)} = 0$.

These examples show that when N = 1, there must be N +1 losses in (21) for its discrete sum to approximate the value of the infinite series on the RHS of (20). Thus the number of Bernoulli variables in (21), τ +1, equals k+N+1. It can be shown that there are N + 1 losses and (21) has length k+N+1 Bernoulli variables when k > 1 or N > 1 as well. The random variable, k successes before N+1 losses in a sequence of independent Bernoulli variables, is distributed as a negative binomial, NB(N,p), probability function (pf), where p is the probability of success, here 1/2. The N parameter approximates the shape parameter of the approximating gamma pdf [7,16]. An expression for the shape parameter of the approximating gamma pdf can be obtained in terms of ω, if an expression can be found for N+1 in terms of ω. N+1 is the discrete approximation to the sum of (1-ω) raised to successively higher powers:

$$N + 1 \approx \sum_{t=0}^{\infty} (1 - \omega)^t \tag{22a}$$

which implies that:

$$\alpha \approx \frac{1-\omega}{\omega} \tag{22b}$$

where α is the shape parameter of the approximating gamma pdf. McDonald and Jensen [42] show that the Gini concentration ratio is a monotonically decreasing function in a gamma pdf of its shape parameter:

$$G = \frac{\Gamma\left(\alpha + \frac{1}{2}\right)}{\sqrt{\pi}\ \Gamma(\alpha + 1)} \tag{22c}$$

which, given (22b), means that the Gini concentration ratio of the gamma pdf approximating the OPIP's distribution of wealth is a monotonically increasing function of ω. The OPIP is consistent with its meta-model's proposition that rising skill levels in the labor force reduce the concentration of wealth. In particular, the OPIP implies that most of the reduction in the concentration of wealth has occurred post-Industrial Revolution with the onset of mass education. However, while the OPIP implies that the concentration of wealth falls as skill levels rise in the labor force, the OPIP also implies that a statistic that many social scientists consider a measure of inequality, dispersion, increases as mean wealth rises. The expression for the variance of wealth in the gamma pdf model of wealth in the OPIP is α/λ^2. Since the mean of the OPIP, μ, is exogenously determined and known, λ can be expressed in terms of α and μ, via the expression for the mean of a gamma distributed random variable in terms of the parameters of the gamma:

$$\frac{\alpha}{\lambda} = \mu \tag{22d}$$

The variance, var(x), of a gamma distributed random variable, x, is:

$$\frac{\alpha}{\lambda^2} = var(x) \tag{22e}$$

So in terms of ω and μ, var(x) of the gamma pdf model of the OPIP stationary distribution of x. wealth, is:

$$var(x) = \frac{\omega\mu^2}{1-\omega} \tag{22f}$$

The OPIP implies that while hunter/gatherer society may appear egalitarian in the sense of a small dispersion of wealth because there is little wealth, hunter/gatherer society has the hottest competition, largest ω, and if it acquires wealth, the greatest concentration of it. According to the OPIP the egalitarianism of hunter/gatherer society is only in terms of the dispersion of wealth which is low because there is little of it. Thus the OPIP is consistent with its meta-model's explanation of the universality of the appearance of inequality when hunter/gatherers acquire food surpluses, wealth. An increase in μ when it is very small, in the presence of large ω, increases the dispersion of wealth and allows the concentration of wealth to be noticed.

6 The Inequality Process with Distributed Omega (IPDO), the OPIP with a Population of Particles each with its ω_i[7]

Figure 1 shows that the shapes of the relative frequency distributions of annual wage and salary income vary by people's level of education in the U.S. Figure 2 shows that these shapes have been stable over 40 years. Figure 3 suggests that these shapes are not peculiar to the U.S. Following its meta-model, the Inequality Process models a labor force composed of workers with different educations via a population of particles each with a possibly distinct value of

[7] Based on Angle [13, 16, and 21].

the ω parameter, i.e. particle i has ω_i, the Inequality Process with Distributed Omega (IPDO). The IPDO models change in the distribution of education in the labor force by change in the proportion, w_ψ, of the population of particles in each ω_ψ equivalence class. When particle i is in the ω_ψ equivalence class, its parameter is referred to as $\omega_{\psi i}$. The meta-model of the Inequality Process implies that $(1-\omega_\psi)$ estimated from observations on the wage incomes of workers at different levels of education should increase at each successively higher level of educaton. Since the effect of education on people's productivity is quasi-permanent during their work lives, education is modeled as time invariant from the point of view of wage income determination. The equations for an encounter between particle i that loses an $\omega_{\psi i}$ share when it loses and particle j that loses an $\omega_{\theta j}$ share when it loses are:

$$
\begin{aligned}
x_{it} &= x_{i(t-1)} + \omega_{\theta j} d_{it} x_{j(t-1)} - \omega_{\psi i}(1-d_{it}) x_{i(t-1)} \\
x_{jt} &= x_{j(t-1)} - \omega_{\theta j} d_{it} x_{j(t-1)} + \omega_{\psi i}(1-d_{it}) x_{i(t-1)}
\end{aligned}
\qquad (23a,b)
$$

where the quantities are defined as in (2a,b) except for:

$\omega_{\psi i}$ = proportion of wealth lost by case i when it loses
$\omega_{\theta j}$ = proportion of wealth lost by case j when it loses.

(23a,b) defines the IPDO.

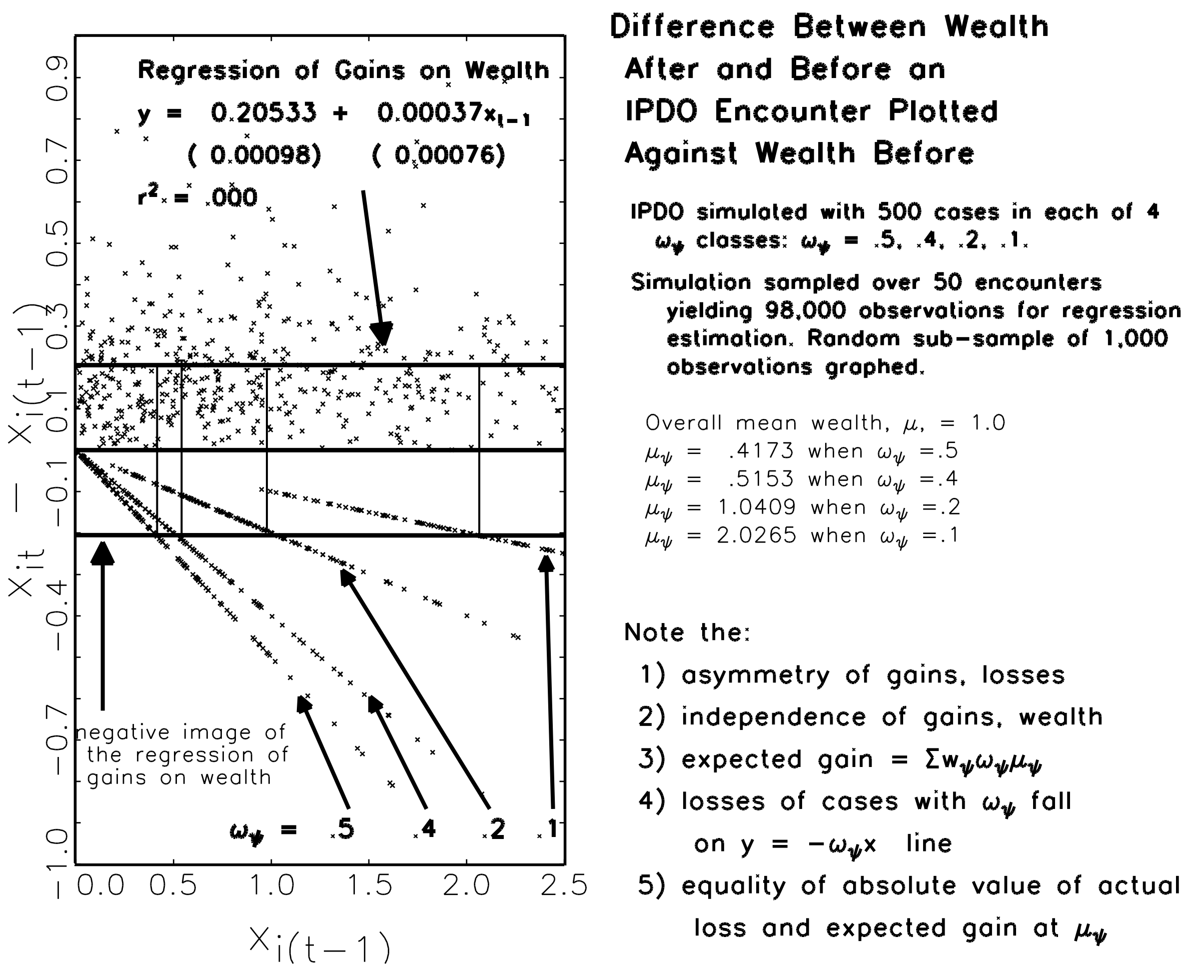

Figure 6: Scattergram of forward differences in the IPDO against wealth

When particle i, in the ω_ψ equivalence class loses, its loss is in absolute value:

$$\omega_{\psi i}\, x_{\psi i(t-1)} \tag{24}$$

See figure 6, the graph of foward differences, x_{it}-$x_{i(t-1)}$, against wealth, $x_{i(t-1)}$ in the IPDO (23a,b). Losses of particles in the ω_ψ equivalence class fall on the line y = $-\omega_\psi x_{\psi i(t-1)}$. When particle i whose parameter is $\omega_{\psi i}$ wins an encounter with particle j whose parameter is ω_θ, its gain is:

$$\omega_{\theta j}\, x_{\theta j(t-1)} \tag{25}$$

The expected gain of all particles in the IPDO (23a,b) is:

$$\overline{\omega\mu} \;\equiv\; \sum_{\psi=1}^{\Psi} w_\psi\, \omega_\psi\, \mu_\psi \tag{26}$$

when there are Ψ distinct ω_ψ equivalence classes. The expectation of gain of particle i is independent of the amount of its wealth, $x_{\psi i(t-1)}$, resulting in a regression line with near zero slope fitted to gains in figure 6.

(23a,b) is solved by backward substitution:

$$\begin{aligned} x_{it} \;=\; & \omega_{\theta j} x_{j(t-1)} d_{it} \\ & + \omega_{\phi k} x_{k(t-2)} d_{i(t-1)} \left[1-\omega_{\psi i}(1-d_{it})\right] \\ & + \omega_{\xi l} x_{l(t-3)} d_{i(t-2)} \left[1-\omega_{\psi i}(1-d_{it})\right]\left[1-\omega_{\psi i}(1-d_{i(t-1)})\right] \\ & + \ldots\ \ldots\ \ldots \end{aligned} \tag{27}$$

The RHS of eq (27), after the realization of d_{it}'s as 0's or 1's, equals:

$$\begin{aligned} & \omega_{\theta j} x_{j(t-1)} d_{it} \\ & + \omega_{\phi k} x_{k(t-2)} d_{i(t-1)} \left(1-\omega_{\psi i}\right)^{1-d_{it}} \\ & + \omega_{\xi l} x_{l(t-3)} d_{i(t-2)} \left(1-\omega_{\psi i}\right)^{2-d_{it}-d_{i(t-1)}} \\ & + \ldots\ \ldots\ \ldots \end{aligned} \tag{28}$$

(28) is the sum of "bites" taken out of competitors multiplied by $(1-\omega_{\psi i})$ raised to the power of the number of later losses, i.e., ego's current wealth, x_{it}, is what it has won from competitors and did not lose at a later time. When $(1-\omega_{\psi i})$ is small, x_{it} is determined by the length of a consecutive run of wins backward in time. Where $(1-\omega_{\psi i})$ is large, losing is less catastrophic and x_{it} can be considered a run of wins backward in time tolerating some intervening losses.

The RHS of (29a) approximates (28) as (20) approximates (15). Ego's wealth is the sum of its gains from competitors, each gain weighted by $(1-\omega_{\psi i})$ raised to the power of the number of later losses:

$$x_{it} \approx \overline{\omega\,\mu} \begin{bmatrix} d_{it} \\ + \; d_{i(t-1)}(1-\omega_{\psi i})^{1-d_{it}} \\ + \; d_{i(t-2)}(1-\omega_{\psi i})^{2-d_{it}-d_{i(t-1)}} \\ + \; d_{i(t-3)}(1-\omega_{\psi i})^{3-\sum_{\tau=0}^{2} d_{i(t-\tau)}} \\ + \; \ldots\ldots\ldots \end{bmatrix}$$

(29a)

where:

$$\overline{\omega\,\mu} \;\equiv\; w_1\omega_1\mu_1 \;+\; \ldots. \;+\; w_\psi\omega_\psi\mu_\psi \;+\; \ldots. \;+\; w_\Psi\omega_\Psi\mu_\Psi$$

(29b)

and:

$$w_i \;>\; 0$$

$$w_1 \;+\; \ldots \;+\; w_\psi \;+\; \ldots \;+\; w_\Psi \;=\; 1.0$$

$$w_\psi \;=\; \frac{n_\psi}{n}$$

(29c)

where n_ψ is the size of the population in the ω_ψ equivalence class, and:

$$n \;=\; n_1 \;+\; n_2 \;+\; \ldots \;+\; n_\psi \;+\; \ldots \;+\; n_\Psi \qquad (29d)$$

Numerically, (29a) approximates (28) better when ω_θ, ω_ϕ, ω_ζ,...., ω_Ψ are small.

The series in brackets on the RHS of (29a) is the series in brackets on the RHS of (20), so the wealth of members of the ω_ψ equivalence class of IPDO particles is approximately distributed as a gamma pdf with the shape parameter:

$$\alpha_\psi \;\approx\; \frac{1-\omega_\psi}{\omega_\psi}$$

(30)

where α_ψ is the shape parameter of the approximating gamma pdf for $\omega_\psi < .5$. Because the run-like series in brackets on the RHS of (29a) is the series inside the parentheses on the RHS of (20), whose expectation is $1/\omega$, mean wealth of the set of cases with ω_ψ, μ_ψ, is:

$$\mu_\psi \;\approx\; \frac{\overline{\omega\mu}}{\omega_\psi}$$

(31)

Given (31) and the fact that the mean of the approximating gamma pdf, μ_ψ, is α_ψ/λ_ψ:

$$\mu_\psi \;\approx\; \frac{\overline{\omega\mu}}{\omega_\psi} \;\approx\; \left(\frac{1\;-\;\omega_\psi}{\omega_\psi}\right)\frac{1}{\lambda_\psi}$$

(32)

which implies:

$$\lambda_\psi \approx \frac{1 - \omega_\psi}{\overline{\omega\mu}}$$

(33)

(29b) defines $\overline{\omega\mu}$ in terms of w_ψ's and ω_ψ's which are known and μ_ψ's which are not. λ_ψ can be solved for in terms of knowns, ω_ψ, w_ψ, and the grand mean, μ, also known, in the following way:

$$\mu = w_1\mu_1 + w_2\mu_2 + \ldots + w_\Psi\mu_\Psi$$

(34)

and from (31):

$$\mu \approx \left(\frac{w_1}{\omega_1}\right)\overline{\omega\mu} + \left(\frac{w_2}{\omega_2}\right)\overline{\omega\mu} + \ldots + \left(\frac{w_\Psi}{\omega_\Psi}\right)\overline{\omega\mu}$$

(35)

which implies that:

$$\overline{\omega\mu} \approx \frac{\mu}{\left(\frac{w_1}{\omega_1} + \frac{w_2}{\omega_2} + \ldots + \frac{w_\Psi}{\omega_\Psi}\right)}$$

(36)

so the RHS of (33) can be expressed in terms of known quantities:

$$\lambda_\psi \approx \frac{1 - \omega_\psi}{\overline{\omega\mu}} \approx \frac{(1-\omega_\psi)\left(\frac{w_1}{\omega_1} + \frac{w_2}{\omega_2} + \ldots + \frac{w_\Psi}{\omega_\Psi}\right)}{\mu} = \frac{1 - \omega_\psi}{\tilde{\omega}\mu}$$

(37)

where $\tilde{\omega}$ is the harmonic mean of the ω_ψ's.

Given (31):

$$\overline{\omega\mu} \approx \omega_\psi\mu_\psi$$

(38)

a loss occurring to an ω_ψ class particle at the conditional mean, the μ_ψ, approximately equals expected gain, $\overline{\omega\mu}$. See the vertical lines in figure 6 at the conditional means, μ_ψ's. The length of the vertical line segment above the x-axis, $\overline{\omega\mu}$, approximately equals the length of the vertical line segment below the x-axis, $\omega_\psi\mu_\psi$. $\overline{\omega\mu}$ can be estimated over any range of income sizes, in particular close to the income size that the definitions and collection practices of large-scale household surveys are optimized for: the median. Estimating $\overline{\omega\mu}$ from gains does not require the identification of the ω_ψ of particles. $\overline{\omega\mu}$ can be estimated either as the intercept of the linear regression of gains on wealth or as the mean gain of particles with a gain. If the ω_ψ' s are known, $\overline{\omega\mu}$ can be estimated as the mean loss (in absolute value) of cases in the ω_ψ equivalence class or as the actual loss at the conditional mean, μ_ψ . Given ω_ψ, an estimate of $\overline{\omega\mu}$ can be used to estimate μ_ψ, or given μ_ψ, $\overline{\omega\mu}$ can be used to estimate ω_ψ.

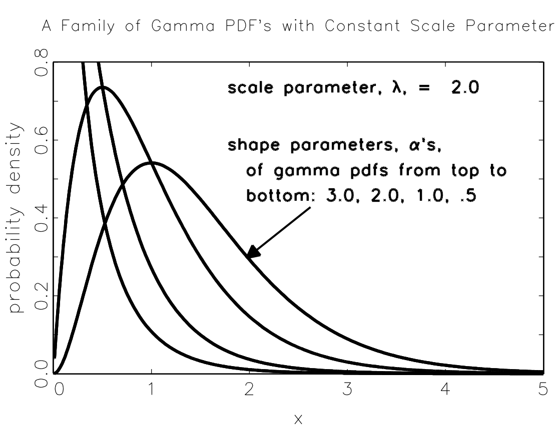

Figure 7: Gamma pdfs with common scale parameter, λ = 2.0, and different shape parameters

An upward shift in the skill levels of the population, resulting by hypothesis in a smaller $\tilde{\omega}_t$ (subscripted t to indicate that it can change over time as the proportion of ω_ψ's change in the population of particles), increases the scale parameter of the approximating gamma pdfs of all ω_ψ equivalence classes, $\lambda_{\psi t}$. A larger $\lambda_{\psi t}$ compresses the distribution of wealth of particles in that equivalence class to the left, decreasing all wealth amounts, as in figure 7, which doubles the scale parameter of figure 5. Compare figure 7 with figure 5. A smaller λ_ψ stretches the mass of the distribution to the right over larger wealth amounts. See figure 8.

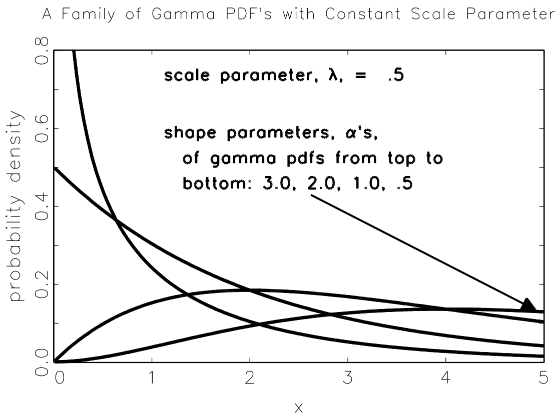

Figure 8: Gamma pdfs with common scale parameter, λ = .5, and different shape parameters

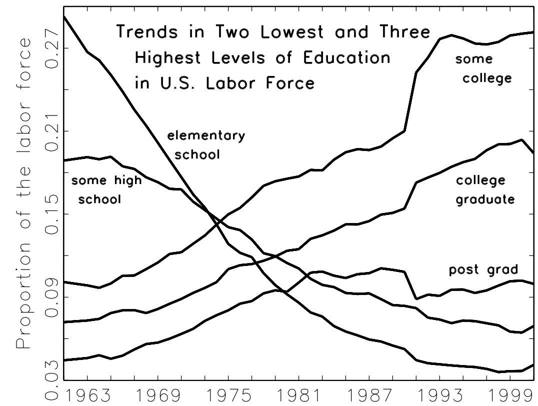

Figure 9
Source: Author's estimates from March CPS data

7 A Test of the Inequality Process with Distributed Omega (IPDO)[8]

The Inequality Process with Distributed Omega (IPDO) (23a,b) can be tested by fitting the gamma pdf model (30), (37) of its stationary distribution of wealth to the distribution of wage income conditioned on education in the U.S., 1961-2001. Education levels in the U.S. labor force rose rapidly during this period putting the IPDO which models the effect of education on the wage distribution to a severe test. See figure 9. The test is the fitting of the gamma pdf model of the IPDO's stationary distribution to each of the partial distributions of the distribution of wage income conditioned on education simultaneously. The fitted model is $f_{\psi t}(x)$, defined by:

$$f_{\psi t}(x) \equiv constant \cdot x^{\alpha_\psi - 1} e^{-\lambda_{\psi t} x}$$
$$\equiv constant \cdot \exp\left(\left(\frac{1-2\omega_\psi}{\omega_\psi}\right)\ln(x) - \left(\frac{1-\omega_\psi}{\mu_t \tilde{\omega}_t}\right)x\right) \tag{39}$$

where the variables are as defined in (23a,b), (30) and (37). (39) is not an exact expression for the stationary distribution of the IPDO (23a,b). Rather, (39) is derived as an approximation to it under the assumption that $\tilde{\omega}_t$ is sufficiently small to justify the gamma pdf approximation, (30), (37), and (39), to the stationary distribution of the IPDO. The expression for the shape parameter, (30), does not change over time. Change enters (39) in the scale parameter, $\lambda_{\psi t}$, via the product $\tilde{\omega}_t\mu_t$ in (37). Fitting (39) to a wage income stream of workers at a particular level of education implicitly capitalizes their wage income stream (or, conversely, annuitizes wealth in the model). μ_t is the unconditional mean of wage income. It is hypothesized, given the meta-model of the Inequality Process, that $\tilde{\omega}_t$ decreases as the level of education rises. Nothing in the model constrains estimated (1- ω_ψ) to vary with education.

Equation (39) is fit to the wage income distribution, expressed in constant 2001 U.S. dollars, of workers 25+ in age at each of six levels of education estimated from 41 years of March Current Population Survey, 1962-2002 data. See Appendix A. There is a simultaneous fit of (39) to each of the 246 partial distributions of wage income using 6

[8] Partially based on Angle [18, 20]

parameters, one ω_ψ for each level of education distinguished. By contrast it would take the estimation of 492 parameters to fit a two parameter gamma pdf to each of the 246 partial distributions. Fitting a two parameter gamma pdf to each of these distributions is a conventional method of modeling them. The greater parsimony of the IPDO's model, (39), is stark. Each partial distribution is estimated by a relative frequency distribution of nine bins resulting in 9 x,y pairs to be fitted, where x = an annual wage and salary income and y = the relative frequency of that x. There is an x,y pair for each relative frequency bin. The bins go from$1-$10,000 to $80,001-$90,000 in constant 2001 dollars. There are 2,214 observations fitted.

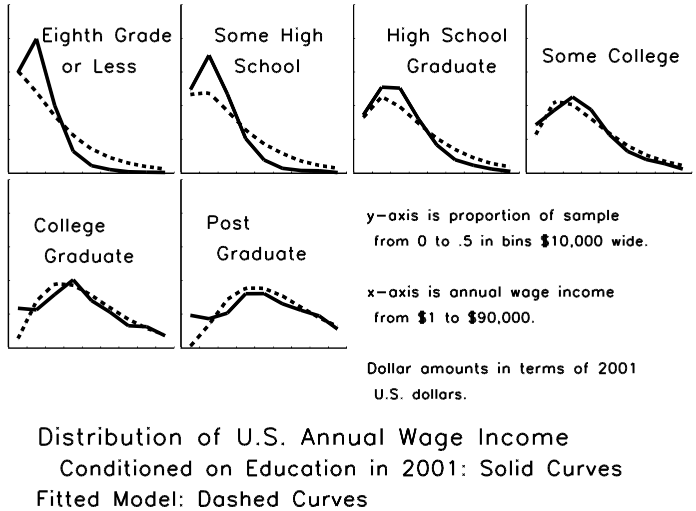

Figure 10: 2001, Year of Worst Fit of Model (39)
Source: Author's estimates from March CPS data

Fitting (39) requires an estimate of the unconditional mean of annual wage and salary income, μ_t. This is obtained from sample estimates of the conditional medians, $x_{(50)\psi t}$:

$$\mu_t \approx \sum_{\psi=1}^{\Psi} w_{it}\, x_{(50)\psi t} \left[\frac{(1 - \omega_\psi)}{\left(1 - \frac{4}{3}\omega_\psi\right)} \right] \qquad (40)$$

where $x_{(50)\psi t}$ is a sample estimate of the ψ^{th} conditional median. See Appendix B for the derivation of (40). μ_t cannot be directly estimated from the sample because of the masking of large incomes in March CPS public use samples.

The model (39) was fitted by nonlinear least squares in which the sum of squared errors, weighted by the w_{it}'s, the proportion of the sample at each education level in each year, is minimized simultaneously for all years by a stochastic search algorithm, a variety of simulated annealing (Kirkpatrick, Gelatt, and Vecchi [47]). The squared correlation between expected relative frequencies under the fitted model and observed sample relative frequencies is .921. As shown in table 1, the estimated $(1-\omega_\psi)$'s scale with education, as the meta-model of the Inequality Process hypothesized. Standard errors are estimated by 100 bootstrap samples. Figure 10 shows the expected relative frequencies under the fitted model (39) and observed relative frequencies in the year of the worst fit of the model. For comparison, figure 11 shows the year of best fit of the model.

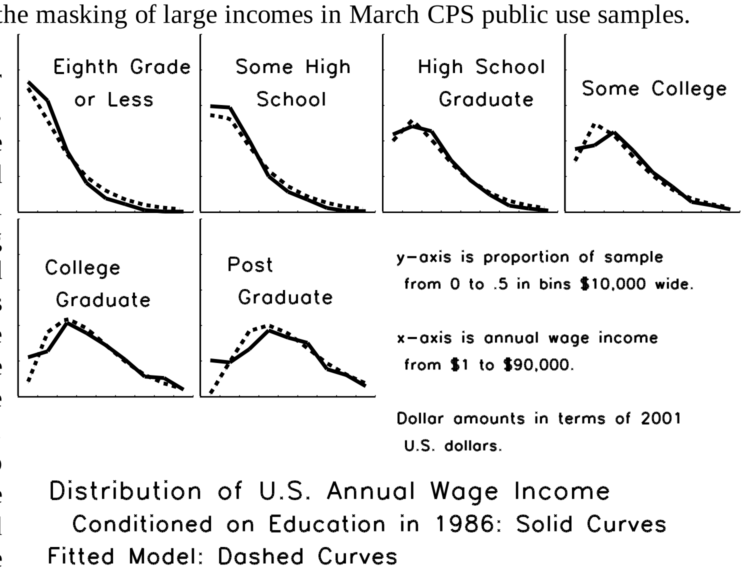

Figure 11: 1985, Year of Best Fit of the Model (39)
Source: Author's estimate from data of the March CPS

Table 1. Estimates of IPDO Model's Parameters

Highest Level of Education	ω_i Estimated by Fitting Model to 246 partial distributions (41 years X 6 levels of education)	Bootstrapped standard error of ω_i (100 re- samples)	Estimate of α_i corresponding to ω_i
Eighth Grade or Less	0.4506	.000098	1.2194

Some High School	0.4005	.000063	1.4972
High School Graduate	0.3554	.000037	1.8134
Some College	0.3255	.000069	2.0718
College Graduate	0.2579	.000186	2.8771
Post Graduate Education	0.2113	.000108	3.7329

In the alternative model, 246 unconstrained two parameter gamma pdfs (492 parameters to be estimated), are each fitted, one at a time. The 2,214 expected frequencies of these 246 fits have a squared correlation with the observed relative frequencies of .943. So the IPDO (39) fits almost as well, at a cost in parsimony of 6 parameters estimated, as the conventional alternative model at a cost of 492 parameters estimated. This test shows that the IPDO's model (39) accounts with great parsimony for the distribution of wage income conditioned on education in the U.S. 1961-2001.

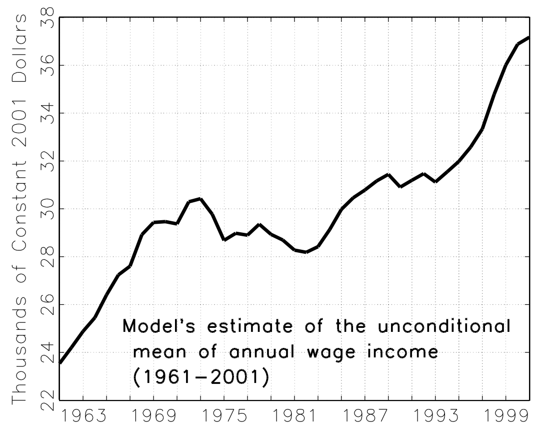

Figure 12: Unconditional mean of wage income in year t, μ_t, 1961-2001, in thousands of constant 2001 dollars estimated by fit of model (39) to March CPS data

Figure 12 shows an estimate of the unconditional mean of annual wage income, μ_t, made via the fitting of the IPDO's model (39). A glance at figure 12 reveals that the unconditional mean of annual wage income increased from year to year most years between 1961 and 2001, although it decreased in some years. Consequently, according to (37), the rising unconditional mean should have contributed to the stretching of the distribution of annual wage income to the right 1961-2001 by making $\lambda_{\psi t}$ smaller. However, as figure 9 shows, more educated workers became a larger part of the labor force in those years. Figure 13 shows a steadily decreasing $\tilde{\omega}_t$, the harmonic mean of the ω_i's, as hypothesized because of the rising level of education in the labor force.

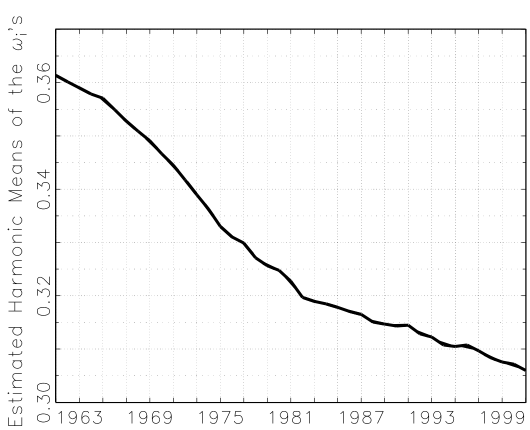

Figure 13: Harmonic mean of ω_i's in year t, $\tilde{\omega}_t$, 1961-2001, estimated by fit of model (39) to March CPS data

In the model if the product $(\tilde{\omega}_t\mu_t)$ increases, the percentiles of the distribution of wealth of each ω_ψ equivalence class increase as the distribution stretches to the right, as in the comparison of figure 8 to figure 5. All percentiles of this distribution increase with increasing $(\tilde{\omega}_t\mu_t)$. Vice versa for a decrease in $(\tilde{\omega}_t\mu_t)$ as the mass of the distribution is compressed to the left and all percentiles decrease, as in the comparison of figure 7 to figure 5. $\tilde{\omega}_t$ decreased while μ_t increased between 1961 and 2001. Figure 14 shows that in the 1960's and 1990's the product $(\tilde{\omega}_t\mu_t)$ trended upward, and also in part of the 1980's. The effect of decreasing $\tilde{\omega}_t$ (due to higher levels of education in the labor force) was to decrease the size of wage gains and exaggerate decreases in wages, i.e., some of the wage gains at every level of education were eaten up by more highly educated workers entering the labor force or less well educated workers leaving it.

8 The Inequality Process with Distributed Omega (IPDO) as a Wealth Maximizing Process

The Inequality Process' meta-model implies that workers who produce more wealth are robust losers in a competition in which all workers are as likely to lose as win. This paper tests and confirms this hypothesis. An empirical process like the IPDO may acquire information about worker productivity from the proportion of wage income lost when a worker loses income. More productive workers may be treated more gently; they may recover more quickly from losses, or it may be that worker skill is a form of wealth less easily expropriated than more tangible forms. By transferring wealth to such workers, this empirical process acts against the scattering of wealth by randomly decided competitions in which the expected difference of wealth after the competitive encounter is less than before (4), and more simply than Maxwell imagined the Demon [48] acting against the scattering of kinetic energy in the ideal gas model. The IPDO's asymmetric treatment of gain and loss accomplishes this feat. The IPDO's doing so via the transfer of wealth to robust losers is counter-intuitive in Western cultures where runs of wins are taken as evidence of merit. Since the SWM's asymmetry between particle gains and losses is obscured by its stochastic driver of wealth exchange, it is not a model that would be naturally selected if robust losers are more productive. An empirical process transferring wealth to robust losers would be naturally selected if a) more productive workers experience smaller losses, b) transferring wealth to the more productive increases the aggregate production of wealth, and c) populations with a greater aggregate production of wealth are selected. Points b and c seem plausible. Point a) should include evidence that more educated workers do not experience smaller losses because of arbitrary discrimination in their favor.

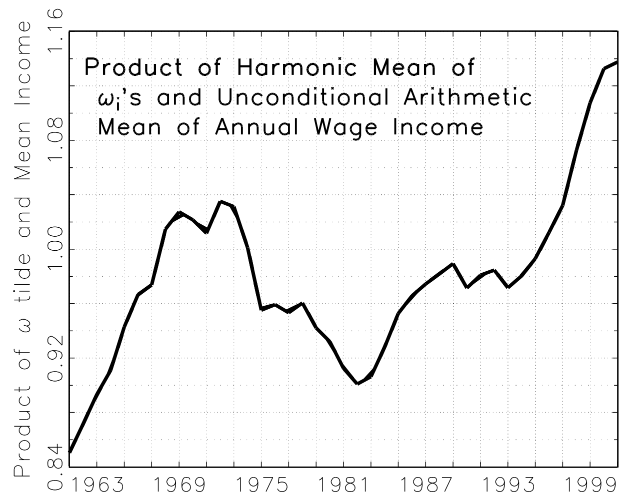

Figure 14: Product ($\tilde{\omega}_t \mu_t$), 1961-2001, estimated by fit of model (39) to data of March CPS

A direct test of point a) requires longitudinal data on worker earnings, a measure of productivity other than earnings or education, and a way of estimating the mean length of time between IPDO particle encounters in terms of of the time units in the data set, an issue discussed in [16]. Such a data set is not at hand. However, there is evidence for points a), b), and c) in the 20th century history of how technology adoption in U.S. agriculture affected farm incomes. The evidence is the falling ratio of mean wealth in the ω_ψ equivalence class, $\mu_{\psi t}$, to the unconditional mean, μ_t, as $\tilde{\omega}_t$ falls. Falling $\tilde{\omega}_t$ is the IPDO's measure of increasing productivity in the population of particles. The relationship between $\mu_{\psi t}$ and μ_t as a function of $\tilde{\omega}_t$ is, given (32) and (37):

$$\mu_{\psi t} \approx \frac{\alpha_\psi}{\lambda_{\psi t}} \approx \frac{1-\omega_\psi}{\omega_\psi} \cdot \frac{\tilde{\omega}_t \, \mu_t}{(1-\omega_\psi)} = \frac{\tilde{\omega}_t \, \mu_t}{\omega_\psi} = \left(\frac{\tilde{\omega}_t}{\omega_\psi} \right) \mu_t \tag{41}$$

Given fixed ω_ψ's, $\mu_{\psi t}$ becomes a smaller fraction of μ_t in every ω_ψ equivalence class as $\tilde{\omega}_t$ falls. With fixed ω_ψ's, $\tilde{\omega}_t$ falls as the proportion of particles in equivalence class ω_θ grows at the expense of the proportion of particles in equivalence class ω_ψ where $\omega_\psi > \omega_\theta$. The scale parameter of the gamma pdf approximation to distribution of wealth in the IPDO, $\lambda_{\psi t}$, is from (37):

$$\lambda_{\psi t} \approx \frac{1 - \omega_\psi}{\tilde{\omega}_t \, \mu_t} \tag{42}$$

If the proportional decrease in $\tilde{\omega}_t$ is greater than the proportional increase in μ_t, the product ($\tilde{\omega}_t \, \mu_t$) decreases and the distribution of wealth in all ω_ψ equivalence classes is compressed left over smaller wealth amounts, as in the comparison of figure 5 to figure 7 since $\lambda_{\psi t}$ is larger. The variance of wealth in the ω_ψ equivalence class decreases rapidly as ($\tilde{\omega}_t \, \mu_t$) decreases since the variance of the gamma pdf approximation to the stationary distribution of wealth in the ω_ψ equivalence class is:

$$var(x_{\psi t}) = \frac{\alpha_\psi}{\lambda_{\psi t}^2} \approx \left(\frac{1-\omega_\psi}{\omega_\psi}\right) \cdot \left(\frac{(\tilde{\omega}_t\ \mu_t)^2}{(1-\omega_\psi)^2}\right) = \frac{(\tilde{\omega}_t\ \mu_t)^2}{\omega_\psi\ (1-\omega_\psi)} \tag{43}$$

So luck in winning IPDO encounters advances the wealth of the lucky less than when ($\tilde{\omega}_t\ \mu_t$) was larger. If particle i were able to decrease ω_i in order to maintain its expected wealth, it would, in a finite population of particles, lower $\tilde{\omega}_t$, further increasing particle i's need to lower ω_i to maintain its expected wealth. Smaller $\tilde{\omega}_t$ means that the wealth of particle i is more closely tied to ω_i.

In a well known phrase in the history of U.S. agriculture, Cochrane [49] labels the effect of the adoption of new agricultural technologies on U.S. farm incomes in the 20th century a "treadmill". The adoption of new agricultural technology provided early adopters only a transient income benefit as other producers adopt the new technology in an attempt to maintain their revenues. Demand for most agricultural commodities has a low price elasticity [50], so increased output due to the new technology lowers the unit price of the crop. While leaving the population is not an option in the IPDO, the bankruptcy of high cost producers buoyed mean farm income. Cochrane's "treadmill" is the transiency of the advantage of being an early adopter of new technology (a step forward on a treadmill moving backward) and the burden of having to continually adopt new technologies and new scales of production to survive. Consumers reaped the windfall increase of wealth from the fall in the unit prices of agricultural rather than producers [50]. This process was a wealth maximization process from the point of view of the economy.

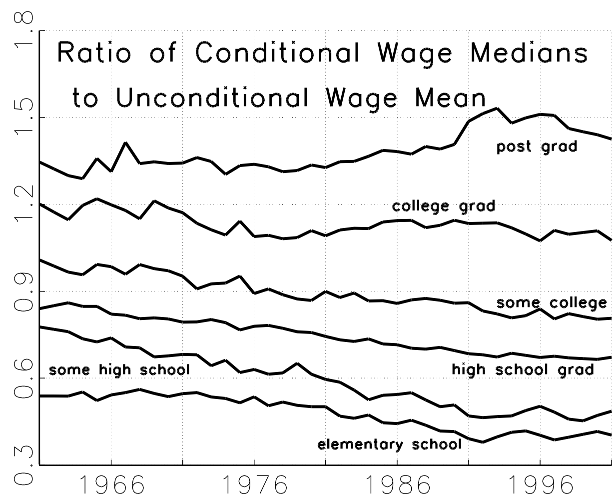

Figure 15: Ratio of sample conditional medians, $x_{(50)\psi t}$, to unconditional mean of wage income, μ_t, estimated by fitting of (39) to distribution of wage income in U.S. conditioned on education, 1961-2001.
Source: Author's estimates based on data of March CPS.

From the perspective of the IPDO, the upgrading of education levels in the U.S. labor force has put the U.S. labor force on the "treadmill" of having to obtain more education to maintain earnings. Figure 15 shows that the effect of a falling $\mu_{\psi t}/\mu_t$ ratio when $\tilde{\omega}_t$ decreases. Figure 15 is the estimated ratio of median of wage income to the unconditional mean of wage income by level of education, 1961 to 2001. Figure 13 shows $\tilde{\omega}_t$ decreasing from about .36 to about .305 in this time period as the level of education of the U.S. labor force rose. The unconditional mean of wage income, μ_t, is estimated in the fitting of the gamma pdf model (39) to the data. See figure 12. As noted in Section 7, the means of wage income at each level of education, however, cannot be directly estimated from March CPS sample data. But each conditional mean can be estimated from the statistic that is perhaps the most reliably estimated sample statistic of wage income, the conditional median. Doodson's approximation to the median implies (Appendix B), that the median wage income among workers at the ψ^{th} level of education, $x_{(50)\psi t}$, is:

$$x_{(50)\psi t} \approx \frac{(3\alpha_\psi - 1)}{3\lambda_{\psi t}} \approx \left(\frac{1 - \frac{4}{3}\omega_\psi}{1-\omega_\psi}\right) \cdot \left(\frac{\tilde{\omega}_t}{\omega_\psi}\right) \cdot \mu_t \tag{44}$$

the product of a constant by the conditional mean in (41), so (44) implies that the ratio of the conditional median to the unconditional mean also decreases as $\tilde{\omega}_t$ decreases. Figure 15 shows that this ratio decreased for five of the six education groups in the U.S. labor force from 1961 through 2001. Thus the IPDO explains why the mean and median of wage income of workers at every level of education decreased relative to the unconditional mean of wage income, except those of the most educated, an open category, in which mean education may have risen.

9. Conclusions

But for their respective stochastic drivers of wealth exchange between particles the One Parameter Inequality Process (OPIP) (2a,b) and [2-23] and the Saved Wealth Model (SWM) of Chakraborti and Chakrabarti (1a,b) and [24-30] are isomorphic stochastic binary interacting particle systems. By contrast, their respective meta-models, the set of understandings about the referents, variables, hypotheses, and tests, are different. The meta-model of each determines each model's choice of stochastic driver of wealth exchange between particles. The meta-model of the SWM is the stochastic version of the ideal gas model. The meta-model of the Inequality Process is the Surplus Theory of Social Stratification of economic anthropology, a verbal theory, as extended by Lenski [32]. This theory explains why the introduction of food surpluses transformed the societal type that anthropologists view as the most egalitarian, the hunter/gatherer, into the chiefdom, the societal type they view as the least egalitarian. This metamorphosis occurred in populations far removed in place, time, race and culture. It is one of a few universal propositions of social science. The Surplus Theory asserts that there was a competition process among the hunter-gatherers which would have concentrated wealth if the hunter-gatherers had much wealth to concentrate. The appearance of storable food surpluses, usually due to the acquisition of agricultural technology, injected wealth into the hunter-gatherers' competition process, which concentrated it. Lenski [32] extends the theory to account for the decreasing concentration of wealth at higher techno-cultural stages than the chiefdom, societal types with greater wealth than the chiefdom. Lenski hypothesizes that more skilled workers create greater wealth, become a larger fraction of the labor force with techno-cultural evolution, and are able to retain a greater proportion of the wealth they create. So the interpretation of $(1-\omega)$, the proportion of wealth retained by a particle in a loss to a competitor, as a worker's skill level is intrinsic to the OPIP's meta-model. As $(1-\omega)$ increases, the Gini concentration ratio of the OPIP's stationary distribution decreases. So the OPIP is consistent with its meta-model.

The SWM uses the stochastic driver of the ideal gas model [31] because the SWM is intended as a generalization of that model, which it subsumes as a special case. While the parameter of the SWM, λ, [not to be confused with the scale parameter of a gamma pdf] is named "savings", its empirical referent is unspecified and no hypothesis about λ is tested in the SWM literature. The word 'competition' does not appear in the SWM literature. Where the Inequality Process's meta-model is a rich source of empirical referents and associated hypotheses and tests, the SWM's meta-model, the ideal gas model, is not.

The OPIP's stationary distribution is a Lévy stable distribution that ranges from Pareto pdf attractor near the upper (hotter) bound of its parameter, ω, to a normal (Gaussian) pdf attractor toward its lower (cooler) bound. A gamma pdf model of the OPIP's stationary distribution is suggested by the solution of the OPIP and works well as an approximation for $\omega < .5$, better as ω approaches its lower bound. (30) and (32) are expressions for the parameters of the approximating gamma pdf in terms of ω. As the SWM is a generalization of the ideal gas model, wealth in the SWM is a generalization of kinetic energy. So the image of temperature in the SWM is $(1-\lambda)\mu$ where μ is mean wealth. The OPIP is μ symmetric. The OPIP takes mean wealth as an exogenous variable. The properties of the OPIP depend on its parameter, ω, whose image in the SWM is $1-\lambda$. So while $\omega\mu$ is the image in the OPIP of $(1-\lambda)\mu$ in the SWM, the OPIP properties of the product $\omega\mu$ depend only on ω. Thus a society such as the chiefdom in OPIP perspective is hotter (has a larger ω) than an industrial democracy, even though mean wealth of the latter is much greater than that of the chiefdom. In the OPIP's perspective techno-cultural evolution has been a cooling process even while mean wealth, μ, has increased proportionally faster than ω has decreased proportionally.

The Inequality Process with Distributed Omega (IPDO) (23a,b) is a generalization of the OPIP. In the IPDO each particle may have a unique value of the parameter ω, i.e., particle i has ω_i. This generalization is required to test the Inequality Process on wage income data from a labor force with workers at different skill levels. The heuristic argument for the derivation of a gamma pdf approximation to the stationary distribution of wealth in the OPIP is applicable to the IPDO's stationary distribution of wealth in the equivalence class of particles with ω_ψ, where $\omega_\psi < .5$. The gamma pdf model that approximates the IPDO's stationary distribution in terms of its parameters, (39), fits the distribution of annual wage income conditioned on education in the U.S. from 1961 through 2001 well. With six levels of education distinguished in the labor force of this period, there are 246 (6 levels of education X 41 years) partial distributions of the conditional distribution to be fitted simultaneously by this 6 parameter model (one parameter for each level of education distinguished). The fit of the alternative model, unconstrained two parameter gamma pdfs individually fitted one at a time to each of the 246 distributions requires the estimation of 492 parameters. The IPDO model fits almost as closely as the fits of the 246 unconstrained two parameter gamma pdfs, fits requiring the estimation of 492 parameters. The six (1- ω_ψ)'s estimated from the fit of the IPDO model (39) scale with the level of worker education as hypothesized in the meta-model of the Inequality Process. See table 1. The estimated ω_ψ's are all less than .5, which, in terms of the IPDO, is why a gamma pdf model is useful for modeling wage income distribution.

The Inequality Process' meta-model determines the choice of its stochastic driver of wealth exchange between particles. The stochastic driver in both OPIP and IPDO is a 0,1 discrete uniform random variate. The SWM's stochastic driver of wealth exchange is a continuous uniform random variate with support at [0.0, 1.0]. The asymmetry between winning and losing is intrinsic in the Inequality Process' meta-model which asserts that the proportion of wealth lost by a worker in a loss varies inversely with the ability of that worker to produce wealth. What is random in an encounter between two particles in the OPIP and IPDO is the determination of which one wins. What the loser gives up to the winner is predetermined from the loser's point of view. It is an ω share in the OPIP or ω_i share in the IPDO for particle i when it loses. What the winner receives is, from the winner's point of view, random. This asymmetry of gain or loss requires a 0,1 discrete random variate to determine which of a pair of particles to a competitive encounter is the winner. The rest is determinate: a fixed proportion of the loser's wealth is transferred to the winner. Particle loss or gain has equal probability in the OPIP and IPDO because the model has no information about the competitive abilities of particles, only that they compete.

The asymmetry of winning and losing provides time reversal asymmetry of particle wealth holding. Given a vector of consecutive wealth amounts of a particle in either the OPIP or IPDO, in chronological order or reverse-chronological order, time flows toward smaller wealth amounts of wealth adjacent in time to larger wealth amounts that are a constant fraction of the larger amount. This fraction is (1-ω) in the OPIP, (1-ω_ψ) in the IPDO. If ω is unknown, it can be calculated from the time-series of wealth holding of a single particle of the OPIP population, or in the case of the IPDO, a single particle in each ω_ψ equivalence class of the IPDO. The SWM for $\lambda = 0$, the ideal gas model, has symmetry of gain and loss and no time-reversal asymmetry. The SWM for $\lambda \neq 0$ has partially asymmetric gains and losses. λ determines the degree of asymmetry. If λ is not known, estimating λ may require a long time series of wealth holding. The problem of estimating the asymmetry of gain and loss in the SWM is not recognized or discussed in the SWM literature.

If a competition process arose via natural selection to allocate wealth to workers who lose less when they lose, robust losers, to increase the aggregate production of wealth, the IPDO would be selected over the SWM because the asymmetry of the IPDO's stochastic driver of wealth exchange lets the IPDO operate with less information than the SWM. The IPDO requires only that robust losers be the more productive workers. Then the IPDO transfers wealth to the more productive, nourishing their production of wealth, maximizing aggregate wealth production. The condition that robust losers are more productive is empirically testable. The flow of wealth toward robust losers in the IPDO is a flow against the entropy maximizing scattering of wealth via random, fair competition. The asymmetry of the IPDO does the work of Maxwell's Demon, more simply. The IPDO does not require that each particle's ω_i remain fixed over time. In a population of particles with time variable ω_{it}, the IPDO continually redirects the flow of wealth from particles with larger ω to smaller ω's.

APPENDIX A: Data and Methods

The distribution of annual wage and salary income is estimated with data from the March Current Population Surveys (1962-2002). The March Current Population Survey (CPS) is known as the Annual Social and Economic Supplement to the monthly Current Population Survey. It has a supplementary questionnaire which includes questions on types of income received in the previous calendar year, posed on behalf of the U.S. Bureau of Labor Statistics. One of the types of income asked about on the March Supplement is total wage and salary income received in the previous calendar year. The CPS is conducted monthly by the U.S. Bureau of the Census (Weinberg, Nelson, Roemer, and Welniak, [51]). The CPS has a substantial number of households in its nationwide sample. The model (39) is tested on the population 25 + in age, earning at least $1 in annual wage and salary income. The age restriction to 25+ is to allow the more educated to be compared to the less educated. The data of the March CPS of 1962 through 2002 were purchased from Unicon, inc. (Unicon, inc, [52]; Current Population Surveys, March 1962-2002 [34]), which provides the services of data cleaning and extraction software, along with research on variable definitions and comparability over time. Unicon, inc was not able to find a copy of the March 1963 CPS which contains data on education. Consequently, the distribution of wage and salary income received in 1962 (from the March 1963 CPS), conditioned on education, is interpolated from the 1961 and 1963 distributions (from the 1962 and 1964 March CPS').

All dollar amounts in the March CPS' are converted to constant 2001 dollars using the PCE (personal consumption expenditure) price index numbers form Table B-7 Chain-type price indexes for gross domestic product, **Economic Report to the President**, February 2003 (Council of Economic Advisers, [53]).

The number of persons in the March Current Population Survey in each year and the number of them meeting the criterion for selection are:

March CPS of	Total number of person records in the March Current Population Survey	people, age 25+, who earned at least $1 in previous calendar year
1962	71,745	22,923
1963	54,282	15,147
1964	54,543	23,903
1965	54,516	23,839
1966	110,055	46,656
1967	104,902	45,266
1968	150,913	47,157
1969	151,848	48,088
1970	145,023	46,004
1971	147,189	46,088
1972	140,432	44,143
1973	136,221	43,200
1974	133,282	43,043
1975	130,124	42,424
1976	135,351	43,888
1977	160,799	52,663
1978	155,706	52,255
1979	154,593	52,793
1980	181,488	63,429
1981	181,358	64,108
1982	162,703	57,877
1983	162,635	57,995
1984	161,167	58,049
1985	161,362	59,819
1986	157,661	59,596
1987	155,468	59,603
1988	155,906	60,501
1989	144,687	57,158
1990	158,079	62,883
1991	158,477	62,942
1992	155,796	62,085
1993	155,197	61,331
1994	150,943	59,575
1995	149,642	59,999
1996	130,476	53,358
1997	131,854	54,553
1998	131,617	54,056
1999	132,324	54,659
2000	133,710	55,925
2001	128,821	53,967
2002	217,219	89,200

The measurement of education changed in the CPS after the 1990 Census from a count of years of school completed to a more degree-oriented measure which better assesses the diversity of post-secondary education. The present study reconciles the two categorizations of educational attainment by collapsing both sets of categories to an ordinal polytomy of six categories. The crudeness of this categorization blurs the distinction between the two different categorizations of educational attainment. The categories of highest level of education attained used here are:

elementary school or less
some high school

completed four years of high school
some college
completed four years of post-secondary education
completed more than four years of post-secondary education

The distribution of annual wage income is estimated as a histogram with relative frequency bins that are of fixed length, $10,000 wide (in terms of constant dollars), to facilitate comparison between the more dense left tail and central mass and the less dense right tail of the distribution.

Appendix B: Estimation of the Unobservable μ_t from Sample Conditional Medians, $x_{(50)\psi t}$'s

The unconditional mean μ_t cannot be reliably estimated from reported wage incomes because a) estimation of the mean is not robust against large outlying wage income observations, b) the CPS' sampling frame is not optimized to sample large incomes, and c) the Census Bureau itself believes that there is substantial error in the measurement of large wage incomes [46]. The estimation of the median avoids these problems. Estimates of it are as robust as any sample statistic of annual wage income can be.

It is possible to estimate μ_t in terms of the sample conditional medians, the $x_{(50)\psi t}$'s, and the proportions of the labor force in each category of education, the $w_{\psi t}$'s (34) (37), by using Doodson's approximation formula for the median, (Weatherburn, 1947 [54],cited in Salem and Mount, [39]):

$$\text{mean} - \text{mode} \approx 3\,(\text{mean} - \text{median})$$

of a two parameter gamma pdf [39]:

$$x_{(50)\psi t} \approx \frac{(3\alpha_\psi - 1)}{3\lambda_{\psi t}}$$

Since:

$$\mu_t = w_{1t}\,\mu_{1t} + w_{2t}\,\mu_{2t} + \ldots + w_{it}\,\mu_{it} + w_{6t}\,\mu_{6t}.$$

The mean of a gamma pdf is the ratio of its shape to its scale parameter:

$$\mu_{\psi t} \approx \frac{\alpha_\psi}{\lambda_{\psi t}}$$

So:

$$\mu_{it} \approx \frac{(3 - 3\,\omega_\psi)}{(3 - 4\,\omega_\psi)} \cdot x_{(50)\psi t}$$

and:

$$\mu_t \approx \sum_\psi w_{\psi t}\, x_{(50)\psi t} \left(\frac{1 - \omega_\psi}{1 - \frac{4}{3}\omega_\psi} \right)$$

where $\omega_\psi < 3/4$.

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