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Statistical Inference for Stochastic Dominance
and for the Measurement of Poverty and Inequality

by

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Abstract

We derive the asymptotic sampling distribution of various estimators frequently used to order distributions in terms of poverty, welfare and inequality. This includes estimators of most of the poverty indices currently in use, as well as estimators of the curves used to infer stochastic dominance of any order. These curves can be used to determine whether poverty, inequality or social welfare is greater in one distribution than in another for general classes of indices. We also derive the sampling distribution of the maximal poverty lines (or income censoring thresholds) up to which we may confidently assert that poverty or social welfare is greater in one distribution than in another. The sampling distribution of convenient estimators for dual approaches to the measurement of poverty is also established. The statistical results are established for deterministic or stochastic poverty lines as well as for paired or independent samples of incomes. Our results are briefly illustrated using data for 6 countries drawn from the Luxembourg Income Study data bases.

On étudie les propriétés asymptotiques de plusieurs estimateurs fréquemment utilisés pour ordonner les répartitions de revenus en termes de pauvreté, bien-être social et inégalité. Ces estimateurs incluent les estimateurs de la plupart des indices de pauvreté couramment en usage ainsi que les estimateurs des courbes utiles pour l'inférence de la dominance stochastique de n'importe quel ordre. Ces courbes nous permettent de déterminer si la pauvreté, l'inégalité ou le bien-être social sont plus élevés dans une répartition que dans une autre pour des classes générales d'indices. On étudie aussi la distribution échantillonnale des seuils maximum de pauvreté ou de censure des revenus jusqu'auxquels on peut affirmer sans ambiguïté que la pauvreté ou le bien-être social sont plus élevés dans une répartition de revenus que dans une autre. La distribution échantillonnale d'estimateurs pour l'approche duale à la mesure de la pauvreté est aussi dérivée. Les résultats statistiques s'appliquent à des seuils déterministes ou stochastiques et à des échantillons dépendants ou indépendants. On illustre brièvement nos résultats à l'aide de données sur 6 pays tirées des banques de données du Luxembourg Income Study.

1. Introduction

Since the influential work of Atkinson (1970), considerable effort has been devoted to making comparisons of welfare distributions more ethically robust, by making judgements only when all members of a wide class of inequality indices or social welfare functions lead to the same conclusion, rather than concentrating on some particular index. More recently, (Atkinson (1987), Foster and Shorrocks (1988a,b), and Howes (1993)), it has been pointed out that similar robustness is desirable for poverty measurement. For instance, Sen (1976) criticises the popular headcount and average poverty gap measures for not taking into account the intensity and the inequality of poverty, respectively.

In this paper, we study estimation and inference in the context of inequality, welfare, and poverty orderings. Our main objective is to show how to estimate orderings which are robust over classes of indices and ranges of poverty lines, and how to perform statistical inference on them.[1] Stochastic dominance, at various orders, allows one to infer whether one population has more or less poverty, welfare, or inequality than another, according to specific wide classes of indices, and so we focus on the asymptotic distributions of estimates of the functions in terms of which stochastic dominance is expressed.

In the next section, we define the various indices in which we are interested in terms of the population distributions to which they apply, and we note some of the relations among them. Many of them can be defined in terms of the functions, which we denote $D^s(x)$, used to define stochastic dominance at order s. For poverty indices, we are interested in stochastic dominance only up to some poverty line. Further, when we compare populations, it is often desirable to use different poverty lines for each population. The use of different poverty lines also leads to the definition of "relative" poverty indices, in which poverty gaps are normalised by the poverty line and with which relative equality dominance can be inferred. Another instrument useful in comparisons is the maximum common poverty line up to which there is more poverty in one population than another, according to the different classes of indices we consider. There exist dual approaches to stochastic dominance, often called p–approaches. We mention the recently developed Cumulative Poverty Gap curve in this context (see Shorrocks (1995a)), and extend it to the case in which different poverty lines are used with different populations.

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[1] Work on these lines can be found in, for instance, Beach and Davidson, (1983), Beach and Richmond (1985), Bishop *et al* (1989), Howes (1993), Davidson and Duclos (1997).

In Section 3, we study estimators of these indices, based on samples drawn from the populations, and we derive their asymptotic distributions. In particular, we discuss the statistical consequences of using *estimated* poverty lines. We also provide estimates of the thresholds up to which one population stochastically dominates another at a given order, and of cumulative poverty gap curves. Our results provide as a corollary the distributions of the two most popular classes of poverty indices, both for deterministic and for sample-dependent poverty lines. The first is the class of additive poverty indices, which include the Foster *et al* (1984) indices, which themselves include the headcount and average poverty gap measures, the Clark *et al* (1981), Chakravarty (1983), and Watts (1968) indices. The second is the class of linear poverty indices, which can be expressed as weighted areas underneath CPG curves. Members of that linear class include the poverty indices of Sen (1976), Takayama (1979), Thon (1979), Kakwani (1980), Hagenaars (1987), Shorrocks (1995a), and Chakravarty (1997).

Since our various estimators can be expressed asymptotically as sums of independently and identically distributed variables, our results apply equally either to the case of observations drawn from independent distributions, and to the case in which dependent observations are drawn from a joint distribution. Dependence would arise, for instance, between paired observations of gross and net incomes, or between paired observations of consumption and income, or among observations of the same individual across time when panel data are used.

Finally, in Section 4, we provide a brief illustration of our techniques using cross-country data from the Luxembourg Income Study data bases. Most of the proofs are relegated to the appendix.

2. Stochastic Dominance and Poverty Indices

Consider two distributions of incomes, F_A and F_B, with support contained in the nonnegative real line. We use the term "income" throughout the paper to signify a measure of individual welfare, which need not be money income. Let $D_A^1(x) = F_A(x)$ and

$$D_A^s(x) = \int_0^x D_A^{(s-1)}(y)\, dy, \qquad (1)$$

for any integer $s \geq 2$, and let $D_B^s(x)$ be defined analogously. It is easy to check inductively that we can express $D^s(x)$ for any order s as

$$D^s(x) = \frac{1}{(s-1)!} \int_0^x (x - y)^{s-1}\, dF(y). \qquad (2)$$

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Distribution B is said to dominate distribution A stochastically at order s if $D_A^s(x) \geq D_B^s(x)$ for all $x \in \mathbb{R}$. For strict dominance, the inequality must hold strictly over some interval of positive measure. Suppose that a poverty line is established at an income level $z > 0$. Then we will say that B (stochastically) dominates A at order s up to the poverty line if $D_A^s(x) \geq D_B^s(x)$ for all $x \leq z$.

First-order stochastic dominance of A by B up to a poverty line z implies that $F_A(x) \geq F_B(x)$ for all $x \leq z$. This is equivalent to the statement that the headcount of individuals below the poverty line is always greater in A than in B for any poverty line not exceeding z.

Second-order dominance of A by B up to a poverty line z implies that $D_A^2(x) \geq D_B^2(x)$, that is, that

$$\int_0^x (x - y) \, dF_A(y) \geq \int_0^x (x - y) \, dF_B(y) \tag{3}$$

for all $x \leq z$. When the poverty line is z, the *poverty gap* for an individual with income y is defined as

$$g(z, y) = (z - y)_+ = \max(z - y, 0) = z - y^* \tag{4}$$

The notation x_+ will be used throughout the paper to signify $\max(x, 0)$. In addition, *censored income* y^* is defined for a given poverty line z as $\min(y, z)$. We can see from (3) that stochastic dominance at order 2 up to z implies that the average poverty gap in A, $D_A^2(x)$, is greater than that in B, $D_B^2(x)$, for all poverty lines x less than or equal to z. The approach is easily generalised to any desired order s.

Ravallion (1994) and others have called the graph of $D^1(x)$ a poverty incidence curve, that of $D^2(x)$ a poverty deficit curve (see also Atkinson (1987)), and that of $D_A^3(x)$ a poverty severity curve. $D^1(x)$ is shown in Figure 1 for two distributions A and B. Distribution B dominates A for all common poverty lines below z. The area underneath $D^1(x)$ for x between 0 and z equals the average poverty gap $D^2(z)$, which is clearly greater for A than for B.

Following Atkinson (1987), we consider the class of poverty indices, defined over poverty gaps, that take the form

$$\Pi(z) = \int_0^z \pi\big(g(z, y)\big) \, dF(y). \tag{5}$$

First, we focus on the class P^1 of indices defined using differentiable, strictly increasing functions π with $\pi(0) = 0$. If we compare two distributions A

and B, then, using definition (4) of the poverty gap, integrating by parts, and changing variables, we find that the difference between Π_A and Π_B for Π in P^1 is given by:

$$\Pi_A(z) - \Pi_B(z) = \int_0^z \pi'(x)\big(F_A(z - x) - F_B(z - x)\big)\,dx \qquad (6)$$

This difference will necessarily be positive for any π with $\pi' > 0$ if and only if

$$F_A(z - x) - F_B(z - x) \geq 0 \qquad (7)$$

for all $x \in [0, z]$ with strict inequality for some interval of x, that is, if B strictly stochastically dominates A at first order up to z. $\Pi_A(x) - \Pi_B(x)$ will then be non-negative for all $x \leq z$ and for all members of P^1.

The headcount index does not fall into the class P^1, because for it π is a constant function equal to unity everywhere. As such it is not strictly increasing, and it does not satisfy $\pi(0) = 0$. However, it is easy to see directly that $\Pi_A - \Pi_B > 0$ for the headcount index if $F_A(z) > F_B(z)$.

We now consider the class of measures P^2 for which π is convex, so that $\pi'' > 0$, and such that $\pi(0) = \pi'(0) = 0$. This is analogous to using social evaluation functions that obey the Dalton principle of transfers (see the discussion of this in Atkinson (1987)). By integrating (6) by parts once more, and noting that the indefinite integral of $F(z - x)$ with respect to x is $-D^2(z - x)$, by (1), we find that

$$\Pi_A(z) - \Pi_B(z) = \int_0^z \pi''(x)\big(D_A^2(z - x) - D_B^2(z - x)\big)\,dx.$$

This will be positive for any strictly convex π if and only if:

$$D_A^2(z - x) - D_B^2(z - x) \geq 0 \qquad (8)$$

for all $x \in [0, z]$ with strict inequality on some subinterval, that is, if B strictly dominates A at second order up to z. $\Pi_A(x) - \Pi_B(x)$ will then be non-negative for all $x \leq z$ and for all members of P^2. It is clear that this sort of reasoning can be extended to any desired order s, thus defining the class P^s of poverty indices, by considering functions π such that $\pi^{(s)}(x) > 0$ for $x > 0$, and $\pi^{(i)}(0) = 0$ for $0 \leq i < s$. A poverty comparison can then be performed by considering the difference $D_A^s(z - x) - D_B^s(z - x)$ over the relevant interval, that is, by examining stochastic dominance at order s up to z.[2]

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[2] For $s = 1, 2$, Foster and Shorrocks (1988b) show how these dominance relationships can be extended to poverty indices (or censored social welfare functions) that may be non-additive.

A useful concept for the analysis of poverty and stochastic dominance is the maximum common poverty line z_s (or censoring point) up to which B stochastically dominates A at order s, so that all the indices in P^s will unanimously indicate that poverty in A is greater than in B if and only if the poverty line z is no greater than z_s.

If B stochastically dominates A for low thresholds z, then either B dominates A everywhere (in which case we have first-order welfare dominance in the sense of Foster and Shorrocks (1988b)), or else there is a reversal at the value z_1 defined by

$$z_1 = \inf \{ x \mid F_A(x) \leq F_B(x) \}. \tag{9}$$

z_1 is illustrated in Figure 1. If z_1 is below the maximal possible income, we can repeat the exercise at order 2. Either B dominates A at second order everywhere,[3] or there exists z_2 defined by

$$z_2 = \inf \{ x \mid D_A^2(x) \leq D_B^2(x) \}. \tag{10}$$

This procedure can be continued until stochastic dominance at some order s is achieved everywhere, or until z_s has become greater than what is seen as a reasonable maximum possible value for the poverty line (or welfare censoring threshold) z. It is shown in Lemma 1 in the Appendix that stochastic dominance of A by B everywhere *will* be achieved for some suitably large value of s.

Comparing poverty across time, societies, or economic environments often involves using different poverty lines for different income distributions. This may be because the relative prices of the goods which must be consumed to maintain a minimum standard of living differ across the distributions, implying a different minimum level of nominal income for one not to be poor, or because nutritional, taste, physiological or climatic factors vary across societies, or simply because the poverty line is deemed to be relative to the distribution of living standards in which the poor happen to live.[4] This last feature is particularly common in studies of poverty in developed economies where a proportion of median or average incomes is often used as a "poverty line" to make cross-country comparisons.[5]

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[3] This is equivalent to Generalised Lorenz dominance of the distribution of incomes in B over that in A, and to second-order welfare dominance.

[4] See, for instance, Greer and Thorbecke (1986) and Ravallion and Bidani (1994), where poverty lines are estimated for different socio-economic groups, and Sen (1981, p.21) on the issue of comparing poverty of two societies with either common or different "standards of minimum necessities".

[5] On this, see, for instance, Smeeding *et al* (1990), Van den Bosch *et al* (1993), Gustafsson and Nivorozhkina (1996), or Atkinson (1995).

We therefore also wish to discuss whether poverty in a distribution A is greater than in a distribution B when separate poverty lines are used for each distribution. We continue to define the classes of poverty indices P^s as above, but now, in order to compare two distributions, A and B, we use two different poverty lines z_A and z_B. In order for there to be at least as much poverty in A than in B according to all indices in the class P^s, it is necessary and sufficient that $D_A^s(z_A - x) - D_B^s(z_B - x) \geq 0$ for all $x > 0$. Of course, this relation no longer constitutes stochastic dominance at order s.

The popular FGT (see Foster *et al* (1984)) class of additive poverty indices is defined by[6]

$$\Delta^\alpha(z) = \int_0^z (z - y)^{\alpha-1} \, dF(y) = \int_0^\infty g(z, y)^{\alpha-1} \, dF(y). \qquad (11)$$

These indices are clearly related to the criteria for stochastic dominance, as was noted by Foster and Shorrocks (1988a,b). In fact, if α is an integer, it follows from (2) that

$$\Delta^\alpha(x) = (\alpha - 1)! \, D^\alpha(x).$$

For any one member of the FGT class of indices, there may be a range of common poverty lines for which poverty in A is greater than in B. For any such line z, the index Δ^s shows more poverty in A than in B if $D_A^s(x) - D_B^s(x) > 0$ for $x = z$, but not necessarily for all $x < z$. Hence, it could be that, for a given range of z, we find dominance of A by B according to Δ^1 and Δ^3, but also find dominance of B by A according to Δ^2, a reversal which would not be possible with stochastic dominance relations. We could then define the thresholds z_s^- and z_s^+, such that B dominates A according to Δ^s only for $z \in [z_s^-, z_s^+]$.

If we use only one member of the FGT class, and are interested only in poverty lines lying in some restricted range, then we need check whether $D_A^s(x) \geq D_B^s(x)$ only for x in that range. For $s = 1$ or $s = 2$, this is the restricted stochastic dominance defined in Atkinson (1987) for the headcount ratio and the mean poverty gap respectively (see his Conditions 1 and 2). It is clear that such restricted dominance conditions can be applied and generalised to any order s of the FGT index.

Other poverty indices can also be expressed in the additive form of (2), that is, as

$$A(z) \equiv \int_0^\infty \delta(y, z) \, dF(y). \qquad (12)$$

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[6] The original FGT indices are normalised by $z^{\alpha-1}$. We return to the interpretation of this normalisation below.

This is the case for the Clark *et al* (1981) second family of indices, for the Chakravarty (1983) index, for which $\delta(y, z) = (y^*)^e$ for $0 < e < 1$, and for the Watts (1968) index, where $\delta(x, y) = \log y^*$. Bourguignon and Fields (1997) also propose an additive index that allows for discontinuities at the poverty line, with $\delta(x, y) = g(z, y)^{\alpha_1 - 1} + \alpha_2 I(y \leq z)$.

Stochastic dominance at first and second order can also be expressed in terms of quantiles. This is called the p–approach to dominance. In order for there to be at least as much poverty in A than in B according to all indices in P^1, it is necessary and sufficient that

$$(z_A - Q_A(p))_+ - (z_B - Q_B(p))_+ \geq 0 \qquad (13)$$

for all $0 \leq p \leq 1$. Here $Q_A(p)$ and $Q_B(p)$ are the p–quantiles of the distributions A and B respectively. If $z_A = z_B$, condition (13) simplifies to checking if the quantiles of B's censored distribution are never smaller than those of A. If this condition is not satisfied for all poverty lines, we may seek the maximum common censoring point z_1 up to which $Q_B(p) - Q_A(p) \geq 0$, for all $p \in [0, 1]$. This is given by z_1 defined in (9) and shown in Figure 1.

There also exists a p–approach to second-order dominance. To see this, define the *cumulative poverty gap* (CPG) curve (also called poverty gap profile by Shorrocks (1995b) and TIP curve by Jenkins and Lambert (1997)) by

$$G(p; z) = \int_0^{Q(p)} g(z, y) \, dF(y). \qquad (14)$$

It is clear that $G(p; z)/p$ is the average poverty gap of the $100p\%$ poorest individuals. Typical CPG curves are shown in Figure 2. For values of p greater than $F(z)$, the CPG curve saturates and becomes horizontal. Since $F(z) = D^1(z)$, the abscissa at which the curve becomes horizontal is the headcount ratio. The ordinate for values of p such that $F(z) \leq p \leq 1$ is readily seen to be $D^2(z)$, the average poverty gap.

To make the link with second-order stochastic dominance, we quote a result of Shorrocks (1995b) and Jenkins and Lambert (1997). They show that, for two distributions A and B and a common poverty line z, it is necessary and sufficient for the stochastic dominance of A by B at second order up to z that $G_A(p; z) \geq G_B(p; z)$ for all $p \in [0, 1]$. The more general case with different poverty lines can be easily derived from Theorem 2 in Shorrocks (1983), if the income distribution in Shorrocks' framework is replaced by the poverty gap distribution in ours. Using Shorrocks' result, we find that poverty is greater in A than in B according to all indices in the class P^2 if and only if the CPG curve for A (using z_A) everywhere dominates the CPG curve for B (using z_B); see also Jenkins and Lambert (1998).

CPG curves can be related to generalised Lorenz curves $GL(p)$, defined by (see Shorrocks (1983)):

$$GL(p) = \int_0^{Q(p)} y \, dF(y).$$

As shown in Figure 2, for $p \leq D^1(z)$, $G(p;z)$ is the vertical distance between the straight line $z \cdot p$ and $GL(p)$. When $p \geq D^1(z)$, $G(p;z)$ is the vertical distance between the line $z \cdot p$ and its tangent to $GL(p)$ at $D^1(z)$. This link between $GL(p)$ and $G(p;z)$ shows that the critical second-order poverty line z_2 defined in (10) equals the slope of the line that is simultaneously tangent to both of the generalised Lorenz curves (at points a and b in Figure 2).

A popular class of poverty measures that are linear in incomes can be easily obtained from $G(p;z)$. To see this, let $\Theta(z)$ measure a weighted area beneath the CPG curve

$$\Theta(z) = \frac{1}{\tau(z) \cdot z} \int_0^{q(z)} \theta(p) \, G(p;z) \, dp \tag{15}$$

$\Theta(z)$ is linear in incomes since $G(p;z)$ is itself a linear (cumulative) function of incomes.[7] Sen's (1976) index is given by setting $\theta(p) = 2$, $\tau(z) = D^1(z)$, and $q(z) = D^1(z)$. $\theta(p) = 2$, $\tau(z) = D^2(z)$ and $q(z) = 1$ yield the Takayama (1979) index. $\theta(p) = 2$, $\tau(z) = 1$ and $q(z) = 1$ give Thon's (1979), Shorrocks' (1995a) and Chakravarty's (1997) poverty indices. Kakwani's (1980) index is obtained with $\theta(p) = \left(k(k+1)\right)\left(D^1(z) - p\right)^{k-1} / (D^1(z))^k$, with $k > 0$, $\tau(z) = 1$, and $q(z) = D^1(z)$. More generally, we can define any linear poverty index $\Theta(z)$ by defining $\theta(p)$ as some particular non-negative function of p. As for the FGT indices, we might also wish to infer the restricted ranges $[z^-, z^+]$ over which the additive or linear indices $A(z)$ and $\Theta(z)$ show more poverty in A than in B.

In the literature on the measurement of poverty, the poverty gap (4), as we remarked above for the case of the FGT indices, is sometimes normalised by the poverty line.[8] We may do this here by replacing absolute poverty gaps $g(z,y)$ by relative poverty gaps $g^r(z,y)$ defined by[9]

$$g^r(z,y) = g(z,y)/z \tag{16}$$

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[7] This is analogous to the definition of the class of linear inequality indices in Mehran (1976).

[8] Although this normalisation is frequently applied, it is not clear that it is desirable when poverty lines differ across groups or societies. For a discussion of this, see Atkinson (1991), p.7 and footnote 3.

[9] For a discussion of absolute versus relative poverty gaps and indices, see Blackorby and Donaldson (1980).

in the definitions of the poverty indices defined in (5). We may further define classes P_r^s of relative poverty indices analogously to the classes P^s, but with $g^r(z, y)$ in place of $g(z, y)$ The stochastic dominance conditions are obviously unchanged if poverty lines are common. By using (16) in (5) and then by successive integrations by parts, we may check that there will be more poverty in A than in B for all indices in P_r^s if and only if

$$\frac{D_A^s(z_A x)}{z_A^{s-1}} - \frac{D_B^s(z_B x)}{z_B^{s-1}} \geq 0 \tag{17}$$

for all $x \in [0, 1]$.

The theoretically equivalent p–approach for class P_r^1 is given by checking whether

$$\frac{\left(z_A - Q_A(p)\right)_+}{z_A} - \frac{\left(z_B - Q_B(p)\right)_+}{z_B} \geq 0. \tag{18}$$

For second-order dominance, the p–approach can be derived by redefining the CPG curve in terms of relative poverty gaps as follows:

$$G^r(p) = \int_0^p \left(\frac{\left(z - Q(q)\right)_+}{z} \right) dq \tag{19}$$

and checking whether $G_A^r(p) - G_B^r(p) \geq 0$ for all $0 < p < 1$.

Finally, for indices for the measurement of relative inequality, we first posit two generally accepted axioms:

Axiom 1: (Atkinson (1970)) When two distributions have the same mean, the rankings in terms of equality dominance are the same as the stochastic or welfare dominance rankings (see also Shorrocks (1983)).

Axiom 2: (Relative equality) Whenever a distribution A can be obtained from a distribution B by multiplying all incomes in B by the same factor $k > 0$, relative income equality in A is necessarily judged to be the same as relative income equality in B (see, *inter alia*, Blackorby and Donaldson (1978)).

The first axiom implies that $D^s(x)$ can be used to check both equality and welfare dominance when means are the same. The second axiom implies that when A and B have different means, we can study equality dominance by comparing the mean-normalised distributions $F_A(x\mu_A)$ and $F_B(x\mu_B)$. This implies checking :

$$\frac{D_A^s(\mu_A x)}{\mu_A^{s-1}} - \frac{D_B^s(\mu_B x)}{\mu_B^{s-1}} \geq 0 \tag{20}$$

for all $x > 0$. For $s = 2$, this is equivalent to checking Lorenz dominance (the Lorenz curve is defined as $L(p) = GL(p)/\mu$).

3. Estimation and Inference

Suppose that we have a random sample of N independent observations y_i, $i = 1, \ldots, N$, from a population. Then it follows from (2) that a natural estimator of $D^s(x)$ (for a nonstochastic x) is

$$
\hat{D}^s(x) = \frac{1}{(s-1)!} \int_0^x (x - y)^{s-1} d\hat{F}(y)
$$

$$
= \frac{1}{N(s-1)!} \sum_{i=1}^N (x - y_i)^{s-1} I(y_i \leq x) \tag{21}
$$

$$
= \frac{1}{N(s-1)!} \sum_{i=1}^N (x - y_i)_+^{s-1}
$$

where $\hat{F}$ denotes the empirical distribution function of the sample and $I(\cdot)$ is an indicator function equal to 1 when its argument is true and 0 otherwise. For $s = 1$, (21) simply estimates the population CDF by the empirical distribution function. For arbitrary s, it has the convenient property of being a sum of IID variables.

When comparing two distributions in terms of stochastic dominance, two kinds of situations typically arise. The first is when we consider two independent populations, with random samples from each. In that case,

$$
\mathrm{var}\big(\hat{D}_A^s(x) - \hat{D}_B^s(x')\big) = \mathrm{var}\big(\hat{D}_A^s(x)\big) + \mathrm{var}\big(\hat{D}_B^s(x')\big). \tag{22}
$$

The other typical case arises when we have N independent drawings of paired incomes, y_i^A and y_i^B, from the same population. For instance, y_i^A could be before-tax income, and y_i^B after-tax income for the same individual i, $i = 1, \ldots, N$. The following theorem allows us to perform statistical inference in both of these cases.

Theorem 1: Let the joint population moments of order $2s - 2$ of y^A and y^B be finite. Then $N^{1/2}(\hat{D}_K^s(x) - D_K^s(x))$ is asymptotically normal with mean zero, for $K = A, B$, with asymptotic covariance structure given by $(K, L = A, B)$

$$
\lim_{N \to \infty} N \, \mathrm{cov}\big(\hat{D}_K^s(x), \hat{D}_L^s(x')\big)
$$

$$
= \frac{1}{\big((s-1)!\big)^2} E\left((x - y^K)_+^{s-1}(x' - y^L)_+^{s-1}\right) - D_K^s(x) \, D_L^s(x').
$$

$$
\tag{23}
$$

Proof: For each distribution, the existence of the population moment of order $s - 1$ lets us apply the law of large numbers to (21), thus showing that

$\hat{D}^s(x)$ is a consistent estimator of $D^s(x)$. Given also the existence of the population moment of order $2s-2$, the central limit theorem shows that the estimator is root-N consistent and asymptotically normal with asymptotic covariance matrix given by (23). This formula clearly applies not only for y^A and y^B separately, but also for the covariance of $\hat{D}^s_A$ and $\hat{D}^s_B$.

If A and B are independent populations, the sample sizes N_A and N_B may be different. Then (23) applies to each with N replaced by the appropriate sample size. The covariance across the two populations is of course zero. ▮

The asymptotic covariance (23) can readily be consistently estimated in a distribution-free manner by using sample equivalents. Thus $D^s(x)$ is estimated by $\hat{D}^s(x)$, and the expectation in (23) by

$$\frac{1}{N}\sum_{i=1}^{N}(x - y_i^K)_+^{s-1}(x' - y_i^L)_+^{s-1}. \qquad (24)$$

In Theorem 1, it was assumed that the argument x of the functions $D^s(x)$ was nonstochastic. In applications one often wishes to deal with $D^s(z-x)$, where z is the poverty line. In the next Theorem, we deal with the case in which z is estimated on the basis of sample information.

Theorem 2: Let the joint population moments of order $2s - 2$ of y^A and y^B be finite. If $s = 1$, suppose in addition that F_A and F_B are differentiable and let $D^0(x) = F'(x)$. Assume first that N independent drawings of pairs (y^A, y^B) have been made from the joint distribution of A and B. Also, let the poverty lines z_A and z_B be estimated by $\hat{z}_A$ and $\hat{z}_B$ respectively, where these estimates are expressible asymptotically as sums of IID variables drawn from the same sample, so that

$$\hat{z}_A = N^{-1}\sum_{i=1}^{N}\xi_A(y_i^A) + o(1) \quad \text{as } N \to \infty, \qquad (25)$$

and similarly for B. Then $N^{1/2}(\hat{D}^s_K(\hat{z}_K - x) - D^s_K(z_K - x))$, $K = A, B$, is asymptotically normal with mean zero, and with covariance structure given by $(K, L = A, B)$

$$\lim_{N \to \infty} N \operatorname{cov}\big(\hat{D}^s_K(\hat{z}_K - x),\ \hat{D}^s_L(\hat{z}_L - x')\big) =$$
$$\operatorname{cov}\Big(D^{s-1}_K(z_K - x)\xi_K(y^K) + \big((s-1)!\big)^{-1}(z_K - x - y^K)_+^{s-1},$$
$$D^{s-1}_L(z_L - x')\xi_L(y^L) + \big((s-1)!\big)^{-1}(z_L - x' - y^L)_+^{s-1}\Big).$$
$$(26)$$

If y^A and y^B are independently distributed, and if N_A and N_B IID drawings are respectively made of these variables, then, for $K = L$, N_K replaces N in (26), while for $K \neq L$, the covariance is zero.

Proof: See appendix. ∎

The sampling distribution of $\Delta^s(x)$ (see (11)) with a fixed x was already derived in Kakwani (1993), Bishop *et al* (1995) and Rongve (1997). The sampling distribution of the headcount when the poverty line is set to a proportion of a quantile is derived in Preston (1995), using results on the joint sampling distribution of quantiles. More generally, the sampling distribution of additive indices when the poverty line is expressed as a sum of IID variables is independently derived in Zheng (1997).

Estimates of the poverty lines may be independent of the sample used to estimate the $D^s(z - x)$, as for example if they are estimated using different data. In that case, the right-hand side of (26) becomes

$$D_K^{s-1}(z_K - x)D_L^{s-1}(z_L - x')\ \mathrm{cov}\big(N^{1/2}(\hat{z}_K - z_K), N^{1/2}(\hat{z}_L - z_L)\big)$$

$$+ \mathrm{cov}\Big(\big((s-1)!\big)^{-1}(z_K - x - y^K)_+^{s-1},\ \big((s-1)!\big)^{-1}(z_L - x' - y^L)_+^{s-1}\Big). \ (27)$$

For indices based on relative poverty gaps, one needs the distribution of $\hat{D}^s(\hat{z}x)$ for positive x; see (17) and (20). The result of Theorem 2 can be used by first eliminating the additive x in that result, and then replacing $\hat{z}$ by $\hat{z}x$.

The covariance (26) can, as usual, be consistently estimated in a distribution-free manner, by the expression

$$N^{-1} \sum_{i=1}^{N} \Bigg(\Big(\hat{D}_K^{s-1}(\hat{z}_K - x)\xi_K(y_i^K) + \big((s-1)!\big)^{-1}(\hat{z}_K - x - y_i^K)_+^{s-1} \Big)$$

$$\times \Big(\hat{D}_L^{s-1}(\hat{z}_L - x')\xi_L(y_i^L) + \big((s-1)!\big)^{-1}(\hat{z}_L - x' - y_i^L)_+^{s-1} \Big) \Bigg)$$

$$- \Bigg(N^{-1} \sum_{i=1}^{N} \Big(\hat{D}_K^{s-1}(\hat{z}_K - x)\xi_K(y_i^K) + \big((s-1)!\big)^{-1}(\hat{z}_K - x - y_i^K)_+^{s-1} \Big) \Bigg)$$

$$\times \Bigg(N^{-1} \sum_{i=1}^{N} \Big(\hat{D}_L^{s-1}(\hat{z}_L - x')\xi_L(y_i^L) + \big((s-1)!\big)^{-1}(\hat{z}_L - x' - y_i^L)_+^{s-1} \Big) \Bigg).$$

The most popular choices of population dependent poverty lines are fractions of the population mean or median, or quantiles of the population distribution. Clearly any function of a sample moment can be expressed asymptotically as an average of IID variables, and the same is true of functions of quantiles, at least for distributions for which the density exists,

according to the Bahadur (1966) representation of quantiles. For ease of reference, this result is cited as Lemma 2 in the Appendix. The result implies that $\hat{Q}(p)$ is root-N consistent, and that it can be expressed asymptotically as an average of IID variables. When the poverty line is a proportion k of the median, for instance, we have that:

$$\xi(y_i) = -k \left(\frac{I\big(y_i < Q(0.5)\big) - 0.5}{F'\big(Q(0.5)\big)} \right),$$

where $Q(0.5)$ denotes the median. When z is k times average income, we have

$$\xi(y_i) = k y_i.$$

This IID structure makes it easy to compute asymptotic covariance structures for sets of quantiles of jointly distributed variables.

We turn now to the estimation of the threshold z_1 defined in (9). Assume that $\hat{F}_A(x)$ is greater than $\hat{F}_B(x)$ for some bottom range of x. If $\hat{F}_A(x)$ is smaller than $\hat{F}_B(x)$ for larger values of x, a natural estimator $\hat{z}_1$ for z_1 can be defined implicitly by

$$\hat{F}_A(\hat{z}_1) = \hat{F}_B(\hat{z}_1).$$

If $\hat{F}_A(x) > \hat{F}_B(x)$ for all $x \leq z$, for some prespecified poverty line z, then we arbitrarily set $\hat{z}_1 = z$. If $\hat{z}_1$ is less than the poverty line z, we may define $\hat{z}_2$ by

$$\hat{D}_A^2(\hat{z}_2) = \hat{D}_B^2(\hat{z}_2)$$

if this equation has a solution less than z, and by z otherwise. And so on for $\hat{z}_s$ for $s > 2$: either we can solve the equation

$$\hat{D}_A^s(\hat{z}_s) = \hat{D}_B^s(\hat{z}_s), \tag{28}$$

or else we set $\hat{z}_s = z$. Note that the second possibility is a mere mathematical convenience used so that $\hat{z}_s$ is always well defined – we may set z as large as we wish. The following theorem gives the sampling distribution of $\hat{z}_s$.

Theorem 3: Let the joint population moments of order $2s - 2$ of y^A and y^B be finite. If $s = 1$, suppose further that F_A and F_B are differentiable, and let $D^0(x) = F'(x)$. Suppose that z_s can be defined by the equation

$$D_A^s(z_s) = D_B^s(z_s),$$

and that $D_A^s(x) > D_B^s(x)$ for all $x < z_s$. Assume that z_s is a simple zero, so that the derivative $D_A^{s-1}(z_s) - D_B^{s-1}(z_s)$ is nonzero. In the

case in which we consider N independent drawings of pairs (y^A, y^B) from one population in which y^A and y^B are jointly distributed, $N^{1/2}(\hat{z}_s - z_s)$ is asymptotically normally distributed with mean zero, and asymptotic variance given by:

$$\lim_{N \to \infty} \text{var}\big(N^{1/2}(\hat{z}_s - z_s)\big) =$$

$$\Big((s-1)!\big(D_A^{s-1}(z_s) - D_B^{s-1}(z_s)\big)\Big)^{-2}$$

$$\Big(\text{var}\big((z_s - y^A)_+^{s-1}\big) + \text{var}\big((z_s - y^B)_+^{s-1}\big) -$$

$$2 \, \text{cov}\big((z_s - y^A)_+^{s-1}, (z_s - y^B)_+^{s-1}\big)\Big).$$

If y^A and y^B are independently distributed, and if N_A and N_B IID drawings are respectively made of these variables, where the ratio $r \equiv N_A/N_B$ remains constant as N_A and N_B tend to infinity, then $N_A^{1/2}(\hat{z}_s - z_s)$ is asymptotically normal with mean zero, and asymptotic variance given by

$$\lim_{N_A \to \infty} \text{var}\big(N_A^{1/2}(\hat{z} - z)\big) = \frac{\text{var}\big((z - y^A)_+^{s-1}\big) + r \, \text{var}\big((z - y^B)_+^{s-1}\big)}{\Big((s-1)!\big(D_A^{s-1}(z) - D_B^{s-1}(z)\big)\Big)^2},$$

Proof: See appendix. ∎

The results of Theorems 1, 2 and 3 can naturally be extended to the additive poverty indices $A(z)$ of (12) by using $A(x)$ in place of $D^s(x)$, $\delta(y, x)$ for $((s-1)!)^{-1}(x - y)_+^{s-1}$, and $A'(x)$ for $D^{s-1}(x)$.

In order to perform statistical inference for p–approaches, we now consider the estimation of the ordinates of the cumulative poverty gap curve $G(p; z)$ defined in (14). The natural estimator, for a possibly estimated poverty line $\hat{z}$, is

$$\hat{G}(p; \hat{z}) = N^{-1} \sum_{i=1}^{N} (\hat{z} - y_i)_+ \, I\big(y_i \leq \hat{Q}(p)\big)$$

where $\hat{Q}(p)$ is the empirical p–quantile. The asymptotic distribution of this estimator is given in the following theorem.

Theorem 4: Let the joint population second moments of y^A and y^B be finite, and let F_A and F_B be differentiable. Let $\hat{z}_A$ and $\hat{z}_B$ be expressible asymptotically as sums of IID variables, as in Theorem 2. If N independent drawings of pairs (y^A, y^B) are made from the joint distribution of A and B, then $N^{1/2}(\hat{G}_K(p; \hat{z}) - G_K(p; z))$, for

$K = A, B$, is asymptotically normal with mean zero, and asymptotic covariance structure given by

$$\lim_{N \to \infty} N \operatorname{cov}\big(\hat{G}_K(p; \hat{z}_K), \hat{G}_L(p'; \hat{z}_L)\big) =$$

$$\operatorname{cov}\Bigg(\Big(I\big(y^K \le Q_K(p)\big)\big((z_K - y^K)_+ - (z_K - Q_K(p))_+\big)$$

$$+ \xi_K(y^K)\,\min\big(p, F_K(z_K)\big)\Big),$$

$$\Big(I\big(y^L \le Q_L(p')\big)\big((z_L - y^L)_+ - (z_L - Q_L(p'))_+\big)$$

$$+ \xi_L(y^L)\,\min\big(p', F_L(z_L)\big)\Big)\Bigg). \qquad (29)$$

If y^A and y^B are independently distributed, and if N_A and N_B IID drawings are respectively made of these variables, then, for $K = L$, N_K replaces N in (29). For $K \ne L$, the covariance is zero.

Proof: See appendix. ∎

If $\hat{z}_A$ and $\hat{z}_B$ are independent of the drawings (y^A, y^B), then the right-hand side of (29) can be modified as in (27).

The arguments used in the proof of Theorem 4 can be used to obtain the asymptotic distribution of all those indices considered in the previous section not already covered by the earlier theorems. First, when z is deterministically set to a level exceeding the highest income in the sample, Theorem 4 yields the sampling distribution of the generalised Lorenz curves, and of the ordinary Lorenz curves when we also take into account the asymptotic distribution of $\hat{\mu}$. Second, for the first-order p–approach, based on quantiles (see (13)), the asymptotic covariance structure is easy to derive because the quantiles can be expressed asymptotically as averages of IID variables, by Bahadur's Lemma, as can the estimated poverty lines, by (25). Third, for the indices based on relative poverty gaps, inference on the expressions in (17), (18), (19) and (20) can be performed by using the asymptotic joint distributions of objects like $\hat{D}^s(x)$, $\hat{z} - \hat{Q}(p)$, $\hat{z}$ and $\hat{\mu}$. Fourth, the asymptotic distribution of estimates $\hat{\Theta}(\hat{z})$ of the linear indices (15) can be readily obtained using the arguments of the proof of Theorem 4. Finally, the asymptotic distribution of estimators of critical poverty lines z (for $z = z^-, z^+$) for the linear indices $\Theta(z)$ can be obtained from Theorem 3 by replacing $(s-1)^{-2} \operatorname{var}\big((z_s - y^K)_+^{s-1}\big)$ by $\lim_{N \to \infty} N \operatorname{var}\big(\hat{\Theta}_K(z)\big)$ and $D_K^{s-1}(z_s)$ by $\Theta'_K(z)$, $K = A, B$.

4. Illustration

We illustrate our results using data drawn from the Luxembourg Income Study (LIS) data sets[10] of the USA, Canada, Italy, the Netherlands, Finland and Norway. These countries were partly selected because of their presence in the 1991 LIS data sets. The raw data were essentially treated in the same manner as in Gottschalk and Smeeding (1997). We take household income to be disposable income (*i.e.*, post-tax-and-transfer income) and we apply purchasing power parities drawn from the Penn World Tables[11] to convert national currencies into 1991 US dollars. As in Gottschalk and Smeeding (1997), we divide household income by an adult-equivalence scale defined as $h^{0.5}$, where h is household size, so as to allow comparisons of the welfare of individuals living in households of different sizes. Hence, all incomes are transformed into 1991 adult-equivalent US\$. All household observations are also weighted by the LIS sample weights "hweight" times the number of persons in the household. Finally, negative incomes are set to 0.

Table 1 shows the estimates of the means and medians of the derived individual income variables for the six countries, along with their asymptotic standard errors. Table 2 shows the headcount $D^1(x)$ for the six LIS countries and for poverty lines varying between US \$2,000 and US \$35,000 (again, in adult-equivalent units). In the following discussion, by "significant" we mean significant at the 5% level. Comparing the US with the other countries, we find that first-order dominance never holds everywhere. Canada has a significantly lower headcount for all x less than or equal to \$30,000 (that is, a poverty line of \$60,000 for a family of 4); in other words, Canada has less poverty than the US for all poverty lines below \$30,000, and for all P^1 poverty indices. The American headcount is significantly lower than Italy's only for those x no less than \$6,000. A similar comment applies to Finland and Norway (\$15,000 for both). As for the Netherlands, its headcount is initially significantly greater than that of the US (for x equal to \$2,000), it is lower than for the US for x between \$4,000 and \$8,000, and it is greater again subsequently.

Table 3 displays similar statistics and results for the average poverty gap $D^2(x)$. The major difference from Table 2 is that Canada now significantly dominates the US for all values of x, and thus we find second-order welfare dominance. As for Italy, Finland and Norway, the initial ranges of x over which they dominate the US are (as expected) greater for $s = 2$ than

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they are in Table 2 for $s = 1$. Compared to the US, the Netherlands have a significantly greater average poverty gap for x equal to \$2,000, a statistically indistinguishable average poverty gap for $x = \$4,000$, a lower one for x between \$6,000 and \$10,000, and a greater average poverty gap for x above \$15,000.

Table 4 shows estimates of the thresholds z_s for dominance relations, and $[z_s^-, z_s^+]$ for restricted dominance relations, between the US and the other five LIS countries, with asymptotic standard errors again shown in parentheses. We find that Canada stochastically dominates the US at the first order up to a censoring threshold of \$27,840, with a standard error on that threshold of \$1,575. As found in the previous table, Canada dominates the US everywhere at the second order. For Italy, Finland and Norway, dominance everywhere is never obtained, even when the order of dominance s increases from 1 to 4, but the critical censoring threshold does move in the expected direction (for Italy, it increases from \$5,340 for $s = 1$ to \$11,042 for $s = 4$, with a standard error relative to the estimates of about 3% to 4%). Looking at the estimates for the comparisons of the Netherlands and of the US, we can conclude that there is first-order dominance of the US for all poverty lines below \$2,958 (with a standard error of \$193), that there is restricted first-order poverty dominance by the Netherlands over the US for poverty lines between \$2,958 and \$8,470, and restricted first-order dominance by the US over the Netherlands for poverty lines above \$8,470 (with a standard error of \$203).

Table 5 presents poverty rankings for the US, Canada and the Netherlands when the poverty line is set to half median income in each country. For $s = 1$, the US has significantly more poverty than in either of the other two countries, whereas the rankings of Canada and the Netherlands switch twice as x approaches 0. For $s = 2$, the USA continues to be dominated (as expected), but poverty then becomes significantly greater in Canada than in the Netherlands for all P^2 indices.

Table 6 confirms the second-order dominance relations of Table 5 by means of the CPG curves for a poverty line set to half median income for the US, Canada and the Netherlands. The CPG point estimates are (as expected) all numerically greatest for the US, followed by those for Canada, and are smallest for the Netherlands. These rankings are everywhere statistically significant, except for values of p equal to or below 0.03.[12]

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[12] At these low values of p, the imprecision associated with the estimation of the p-quantiles is sufficiently large to prevent a clear statistical ordering of the distributions. This precision problem seems less important in the use of the $D^s(x)$ approach, as seen in the previous table. Which of the two approaches is generally more statistically efficient remains, however, a topic for future research.

Tables 7 (for $s = 2$) and 8 (for $s = 3$) present the estimates for the relative equality dominance rankings of Italy, Canada, the US and the Netherlands for values of x ranging between 0.25 and 2.5. For poverty dominance using relative poverty gaps (and when the poverty line is set to mean income), the same estimates can be used, but one then needs to focus on values of x between 0 and 1. In Table 7, it is not possible to rank Canada, Italy and the Netherlands unambiguously in terms of equality: the ranks change with the values of x. The USA is significantly less equal than Canada and Italy for almost all values of x, except when x rises above 2.0.[13] It is, however, possible to infer that the US has more poverty than Canada and Italy for all P_r^2 poverty indices, although for $x = 0.25$, we cannot infer that the US has more poverty than the Netherlands. For $s = 3$ (Table 8), unambiguous equality dominance of Canada and Italy over the US can now be inferred for all values of x, although there is still ambiguity for $x = 0.25$ between the US and the Netherlands. And, even at this third-order level of dominance, ambiguity still remains in the equality rankings of Canada, Italy and the Netherlands.

Appendix

Lemma 1:　　If B dominates A strictly for $s = 1$ up to $w > 0$, then for any finite threshold z, B dominates A at order s up to z for s sufficiently large.

Proof: We have $F_A(x) - F_B(x) > 0$ for $x < w$. Let $a = \min(F_A(x) - F_B(x))$ over the range $[0, w/2]$, say, so that $a > 0$ strictly. We wish to show that, for large s, $D_A^s(x) - D_B^s(x) > 0$ for $x < z$, that is,

$$\int_0^x \left(1 - \frac{y}{x}\right)^{s-1} \left(dF_A(y) - dF_B(y)\right) > 0 \tag{30}$$

for $x < z$. For ease in the sequel, we have multiplied $D^s(x)$ by $(s-1)!/x^{s-1}$, which does not affect the inequality we wish to demonstrate.

Now the left-hand side of (30) can be integrated by parts to yield

$$\frac{s-1}{x} \int_0^x \left(F_A(y) - F_B(y)\right) \left(1 - \frac{y}{x}\right)^{s-2} dy.$$

We split this integral in two parts: the integral from 0 to $w/2$, and then from $w/2$ to x. We may bound the absolute value of the second part: since

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[13]　The values of $D^2(\mu \cdot x)/\mu$ will inevitably coincide for all countries when x is large enough, since $D^2(\mu \cdot x)/\mu$ then always tends to $x - 1$. The existence of this common limiting value is analogous to the convergence to 1 of the Lorenz curve of any distribution as p approaches 1.

$|F_A(y) - F_B(y)| \leq 1$ for any y, we have

$$\left| \int_{w/2}^{x} \big(F_A(y) - F_B(y)\big) \left(1 - \frac{y}{x}\right)^{s-2} dy \right| \leq \left| \int_{w/2}^{x} \left(1 - \frac{y}{x}\right)^{s-2} dy \right|$$

$$= \frac{x}{s-1} \left(1 - \frac{w}{2x}\right)^{s-1}. \qquad (31)$$

The first part is greater than

$$a \frac{s-1}{x} \int_{0}^{w/2} \left(1 - \frac{y}{x}\right)^{s-2} dy = a \left(1 - \left(1 - \frac{w}{2x}\right)^{s-1}\right). \qquad (32)$$

We can make s large enough that, for all $w < x < z$, $(1 - w/2x)^{s-1} < a/3$. Then, by (31) and (32), (30) is greater than $a/3$ for all $w < x < z$. For $x < w$, the dominance at first order up to w implies dominance at any order $s > 1$ up to w. The result is therefore proved. ∎

Lemma 2: (Bahadur, 1966). Suppose that a population is characterised by a twice differentiable distribution function F. Then, if the p–quantile of F is denoted by $Q(p)$, and the sample p–quantile from a sample of N independent drawings y_i from F by $\hat{Q}(p)$, we have

$$\hat{Q}(p) - Q(p) = -\frac{1}{Nf\big(Q(p)\big)} \sum_{i=1}^{N} \Big(I\big(y_i < Q(p)\big) - \alpha\Big) + O\big(N^{-3/4}(\log N)^{3/4}\big),$$

where $f = F'$ is the density.

Proof of Theorem 2:

For distributions A and B, we have

$$(s-1)! \, \hat{D}^s(\hat{z} - x) = \int_{0}^{\hat{z}-x} (\hat{z} - x - y)^{s-1} d\hat{F}(y) \quad \text{and}$$

$$(s-1)! \, D^s(z - x) = \int_{0}^{z-x} (z - x - y)^{s-1} dF(y).$$

Thus

$$(s-1)! \left(\hat{D}^s(\hat{z}-x) - D^s(z-x) \right) =$$

$$\int_{z-x}^{\hat{z}-x} \left((\hat{z}-x-y)^{s-1} - (z-x-y)^{s-1} \right) d\hat{F}(y)$$

$$+ \int_{z-x}^{\hat{z}-x} (z-x-y)^{s-1} d(\hat{F}-F)(y)$$

$$+ \int_{z-x}^{\hat{z}-x} (z-x-y)^{s-1} dF(y) \qquad (33)$$

$$+ \int_0^{z-x} \left((\hat{z}-x-y)^{s-1} - (z-x-y)^{s-1} \right) d(\hat{F}-F)(y)$$

$$+ \int_0^{z-x} \left((\hat{z}-x-y)^{s-1} - (z-x-y)^{s-1} \right) dF(y)$$

$$+ \int_0^{z-x} (z-x-y)^{s-1} d(\hat{F}-F)(y).$$

It follows from (25) that $\hat{z} - z = O(N^{-1/2})$, and by standard properties of the empirical distribution, $\hat{F} - F = O(N^{-1/2})$. Thus the first two terms and the fourth are of order N^{-1}, and the others are of order $N^{-1/2}$.

The third term can be expressed as:

$$\int_{z-x}^{\hat{z}-x} (z-x-y)^{s-1} dF(y) = \int_0^{z-\hat{z}} u^{s-1} dF(z-x-u) = O(N^{-s/2}),$$

from which we see that it contributes asymptotically only if $s = 1$. In that case, the term is

$$F(\hat{z}-x) - F(z-x) = D^0(z-x)(\hat{z}-z) + O(N^{-1}),$$

since we made the definition $D^0 = F'$.

The fifth term is obviously zero for $s = 1$. For $s > 1$, it can be expressed as

$$(\hat{z}-z) \int_0^{z-x} \sum_{k=0}^{s-2} (\hat{z}-x-y)^k (z-x-y)^{s-2-k} dF(y) =$$

$$(\hat{z}-z)(s-1) \int_0^{z-x} (z-x-y)^{s-2} dF(y) + O(N^{-1}) =$$

$$(\hat{z}-z)(s-1)! \, D^{s-1}(z-x) + O(N^{-1}). \qquad (34)$$

We see that expression (34) serves for the fifth term when $s > 1$ and for the third when $s = 1$.

Finally, the sixth term is

$$\int_0^{z-x} (z - x - y)^{s-1} \, d(\hat{F} - F)(y)$$

$$= \frac{1}{N} \sum_{i=1}^{N} \left((z - x - y_i)_+^{s-1} - E\left((z - x - y)_+^{s-1} \right) \right),$$

and so it is the average of N IID variables of mean zero. Multiplying (33) by $N^{1/2}$, we see that

$$N^{1/2} \left(\hat{D}^s(\hat{z} - x) - D^s(z - x) \right) = D^{s-1}(z - x) N^{1/2}(\hat{z} - z)$$

$$+ \frac{1}{(s-1)!} N^{-1/2} \sum_{i=1}^{N} \left((z - x - y_i)_+^{s-1} - E\left((z - x - y)_+^{s-1} \right) \right). \tag{35}$$

The result of the theorem follows from (35) by simple calculation. ∎

Proof of Theorem 3:

Consider the general problem in which, for some population, a value z is defined implicitly by $h(z) = 0$, where the function h is defined in terms of the population distribution. For instance, if $Q(p)$ is the p–quantile of a distribution with CDF F, we have $F(Q(p)) = p$, and we can set $h(z) = F(z) - p$.

For z_s, the defining relationship, in terms of the populations A and B, is $D_A^s(z_s) = D_B^s(z_s)$, with $D_A^s(x) > D_B^s(x)$ for all $x < z_s$. Thus we set $h(x) = D_A^s(x) - D_B^s(x)$. According to (28), $\hat{z}_s$ is defined in terms of $\hat{h}(x) \equiv \hat{D}_A^s(x) - \hat{D}_B^s(x)$. Under the assumption that z_s exists in the population and is less than the poverty line z, $\hat{z}_s$ is clearly a consistent estimator of z_s, and, in particular, we need not consider the possibility that $\hat{z}_s = z$, since this will happen with vanishingly small probability as $N \to \infty$.

The proof is similar for all values of s, and so we drop s from our notations. Since $h(z) = 0$, we have by Taylor expansion that

$$h(\hat{z}) = h'(\tilde{z})(\hat{z} - z) \tag{36}$$

for some $\tilde{z}$ such that $|\tilde{z} - z| < |\hat{z} - z|$. We will show later that

$$\hat{h}(z) + h(\hat{z}) = o(N^{-1/2}). \tag{37}$$

It was assumed that $h'(z) \neq 0$, and, in fact, since $h(x) > 0$ for $x < z$, and $h(z) = 0$, it follows that $h'(z) < 0$. Since $\hat{z} \to z$ as $N \to \infty$, we have that

$\tilde{z} \to z$ as $N \to \infty$ as well. Thus for large enough N, $h'(\tilde{z}) \neq 0$. It follows from (36) and (37) that

$$\hat{z} - z = -\frac{\hat{h}(z)}{h'(\tilde{z})} + o(N^{-1/2}). \qquad (38)$$

Suppose first that the populations A and B are independent, and that we have N_A drawings from one and N_B drawings from the other. For the purposes of the asymptotic analysis, we assume that the ratio $r = N_A/N_B$ remains constant as N_A and N_B tend to infinity. We have that

$$E\left((z - y^A)_+^{s-1}\right) = (s-1)!\, D_A^s(z) = (s-1)!\, D_B^s(z) = E\left(z - y^B\right)_+^{s-1}$$

because $h(z) = 0$. It follows that

$$N_A^{1/2}\hat{h}(z) = \frac{1}{(s-1)!}\left(N_A^{-1/2}\sum_{i=1}^{N_A}\left((z - y_i^A)_+^{s-1} - E\left((z - y^A)_+^{s-1}\right)\right)\right.$$

$$\left. - r^{1/2}N_B^{-1/2}\sum_{j=1}^{N_B}\left((z - y_j^B)_+^{s-1} - E\left(z - y^B\right)_+^{s-1}\right)\right). \qquad (39)$$

The expression (39) consists of two independent sums of IID variables to which we may apply the central limit theorem since moments of order $2s - 2$ are assumed to exist. It follows immediately that $N_A^{1/2}\hat{h}(z) = O(1)$ in probability, and, from (38), that $\hat{z} - z = O(N^{-1/2})$. In addition, from (1),

$$h'(z) = D_A^{s-1}(z) - D_B^{s-1}(z). \qquad (40)$$

If $s = 1$, (40) remains correct because we defined $D_A^0(z) = F_A'(z)$, the density associated with the CDF F_A. We now see from (38) and (39) that

$$\lim_{N_A \to \infty} \text{var}\left(N_A^{1/2}(\hat{z} - z)\right) = \frac{\text{var}\left((z - y^A)_+^{s-1}\right) + r\,\text{var}\left((z - y^B)_+^{s-1}\right)}{\left(\left(D_A^{s-1}(z) - D_B^{s-1}(z)\right)(s-1)!\right)^2}, \qquad (41)$$

Next, suppose that we have N paired observations y_i^A and y_i^B from one single population. (39) continues to hold with $N_A = N$ and $r = 1$. However, the two sums of IID variables are no longer independent in general, and so (41) must be replaced by

$$\lim_{N \to \infty} \text{var}\left(N^{1/2}(\hat{z} - z)\right) =$$

$$\frac{\text{var}\left((z - y^A)_+^{s-1}\right) + \text{var}\left((z - y^B)_+^{s-1}\right) - 2\,\text{cov}\left((z - y^A)_+^{s-1}, (z - y^B)_+^{s-1}\right)}{\left(\left(D_A^{s-1}(z) - D_B^{s-1}(z)\right)(s-1)!\right)^2}.$$

$$(42)$$

It remains to prove (37). Note that, because $\hat{h}(\hat{z}) = h(z) = 0$,

$$-\big(\hat{h}(z) + h(\hat{z})\big) = \hat{h}(\hat{z}) - \hat{h}(z) - \big(h(\hat{z}) - h(z)\big). \tag{43}$$

Consider the expression

$$\hat{h}(z + \delta) - \hat{h}(z) - \big(h(z + \delta) - h(z)\big). \tag{44}$$

for nonrandom δ. In the case of just one population and N paired drawings of y^A and y^B, we can write

$$\hat{h}(z + \delta) - \hat{h}(z) = \frac{1}{N(s-1)!} \Big(\sum_{i=1}^{N} (z + \delta - y^A)_+^{s-1} - (z + \delta - y^B)_+^{s-1}$$
$$- (z - y^A)_+^{s-1} + (z - y^B)_+^{s-1} \Big).$$

The expectation of this is $h(z + \delta) - h(z)$, and so (44) is the average of bounded IID variables with mean zero and finite variance of order δ^2. Consequently, by the central limit theorem, (44) times $N^{1/2}$ has mean zero and variance of order δ^2. Since $\hat{z} - z = O(N^{-1/2})$ in probability, it follows that (43) times $N^{1/2}$ tends to zero in mean square, and hence in probability. An exactly similar argument applies when there are two populations. ∎

Proof of Theorem 4:
We have for both distributions A and B that

$$\hat{G}(p; \hat{z}) = \int_0^\infty (z - y) I(y < \hat{z}) \, I(y < \hat{Q}(p)) \, d\hat{F}(y)$$
$$+ (\hat{z} - z) \int_0^\infty I(y < \hat{z}) \, I(y < \hat{Q}(p)) \, d\hat{F}(y). \tag{45}$$

The second term on the right-hand side of this is

$$(\hat{z} - z)\hat{F}\big(\min(\hat{z}, \hat{Q}(p))\big)$$
$$= (\hat{z} - z) \min\big(\hat{F}(\hat{z}), p\big)$$
$$= (\hat{z} - z) \min\big(F(z), p\big) + O(N^{-1}),$$

and the first term is

$$\int_0^{\hat{Q}(p)} (z - y)_+ \, d\hat{F}(y).$$

This kind of integral can be expressed asymptotically as a sum of IID variables using a technique developed in Davidson and Duclos (1997). The term becomes

$$p\big(z - Q(p)\big)_+ + N^{-1} \sum_{i=1}^{N} I\big(y_i < Q(p)\big)\big((z - y_i)_+ - (z - Q(p))_+\big) + O(N^{-1}),$$

which to leading order is a deterministic term plus an average of IID random variables. We can combine the two terms in (45) using (25) to get

$$\hat{G}(p; \hat{z}) = p(z - Q(p))_+ + N^{-1} \sum_{i=1}^{N} \Big(I\big(y_i < Q(p)\big)\big((z - y_i)_+ - (z - Q(p))_+\big)$$

$$+ \big(\xi(y_i) - z\big) \min\big(F(z), p\big)\Big) + O(N^{-1}). \qquad (46)$$

If z is known and not estimated, we can just set $\xi(y_i) = z$, and the last term in the sum will vanish.

It is easy to check that, whether $z < Q(p)$ or $z > Q(p)$, the expectation of the leading term of the above expression is just $G(p; z)$. The fact that $\hat{G}_A(p; \hat{z}_A)$ and $\hat{G}_B(p; \hat{z}_B)$ are sums of independently and identically distributed random variables with finite second moments leads to their asymptotic normality by the central limit theorem. The covariance structure is obtained by simple calculation. ∎

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Figure 1

Poverty incidence curves
for two distributions A and B
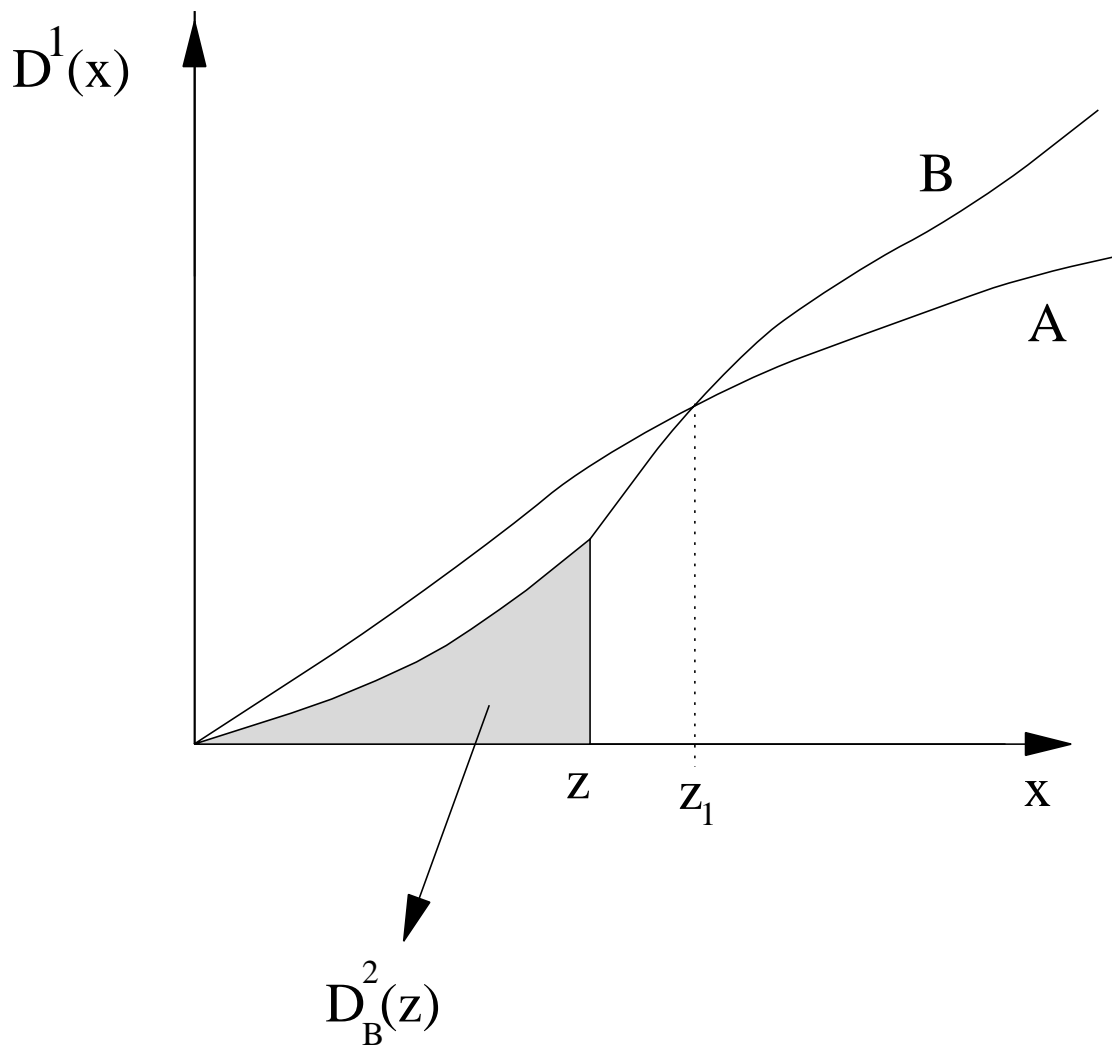
Figure 2

CPG and Generalised Lorenz curves
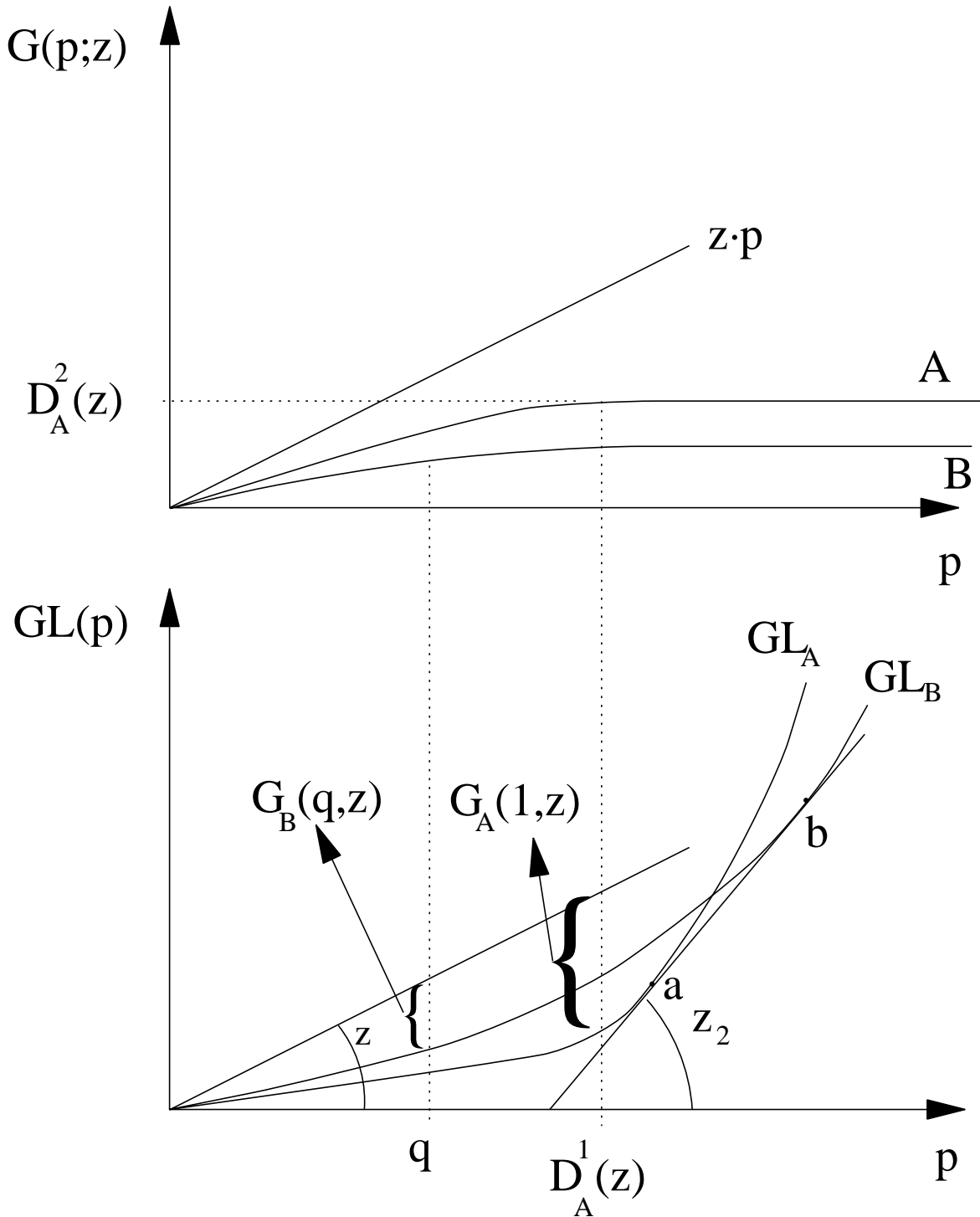
Table 1

Sample means and medians (in 1991 equivalised US dollars)

(asymptotic standard errors)

	USA	Canada	Italy	Finland	Norway	Netherlands
Mean	18589	19807	12699	15182	13902	15820
	(95)	(76)	(82)	(61)	(87)	(137)
Median	16320	17883	11356	14306	12532	14836
	(110)	(77)	(85)	(60)	(79)	(107)
Standard deviation	12098	11231	7440	6574	7881	9058
Number of LIS sample observations	16052	21647	8188	11749	8073	4378

Table 2

Headcounts for various poverty lines

(asymptotic standard errors)

	Estimates of $D^1(x)$					
x	**USA**	**Canada**	**Italy**	**Finland**	**Norway**	**Netherlands**
2000	0.0184 (0.001)	0.0071 (0.0006)	0.0089 (0.001)	0.0031 (0.0005)	0.0059 (0.0009)	0.0234 (0.002)
4000	0.0461 (0.002)	0.0162 (0.0009)	0.0349 (0.002)	0.0096 (0.0009)	0.0122 (0.001)	0.0332 (0.003)
6000	0.104 (0.002)	0.042 (0.001)	0.124 (0.004)	0.034 (0.002)	0.026 (0.002)	0.061 (0.004)
8000	0.176 (0.003)	0.089 (0.002)	0.276 (0.005)	0.091 (0.003)	0.086 (0.003)	0.159 (0.006)
10000	0.250 (0.003)	0.149 (0.002)	0.408 (0.005)	0.184 (0.004)	0.173 (0.004)	0.310 (0.007)
15000	0.451 (0.004)	0.366 (0.003)	0.707 (0.005)	0.553 (0.004)	0.511 (0.006)	0.660 (0.007)
20000	0.625 (0.004)	0.584 (0.003)	0.879 (0.004)	0.830 (0.003)	0.796 (0.004)	0.856 (0.005)
25000	0.761 (0.003)	0.751 (0.003)	0.949 (0.002)	0.936 (0.002)	0.927 (0.003)	0.944 (0.003)
30000	0.854 (0.003)	0.859 (0.002)	0.979 (0.002)	0.972 (0.002)	0.970 (0.002)	0.973 (0.002)
35000	0.908 (0.002)	0.923 (0.002)	0.988 (0.001)	0.987 (0.001)	0.984 (0.001)	0.985 (0.002)

Table 3

Average poverty gaps for various poverty lines

(asymptotic standard errors)

			Estimates of $D^2(x)$			
x	**USA**	**Canada**	**Italy**	**Finland**	**Norway**	**Netherlands**
2000	23.0 (1.5)	8.7 (0.8)	8.6 (1.3)	2.9 (0.6)	8.5 (1.3)	34.7 (3.7)
4000	81.2 (3.7)	31.4 (2.0)	47.4 (3.6)	13.8 (1.7)	27.4 (3.2)	90.0 (8.2)
6000	227.7 (6.9)	86.7 (3.7)	188.7 (7.7)	52.1 (3.5)	64.7 (5.6)	177.4 (13.4)
8000	505.5 (11.1)	216.8 (6.2)	588.8 (14.1)	170.8 (6.7)	172.1 (9.1)	374.8 (19.8)
10000	933 (16)	453 (10)	1274 (22)	439 (11)	429 (14)	843 (28)
15000	2694 (31)	1714 (21)	4109 (42)	2246 (27)	2101 (33)	3313 (54)
20000	5397 (45)	4112 (33)	8120 (56)	5785 (40)	5447 (50)	7153 (74)
25000	8884 (58)	7478 (44)	12714 (64)	10250 (48)	9811 (60)	11702 (85)
30000	12941 (60)	11531 (53)	17539 (69)	15036 (53)	14581 (65)	16506 (92)
35000	17362 (75)	16004 (58)	22461 (72)	19940 (55)	19468 (69)	21410 (97)

Table 4

**Estimates of the thresholds z_s for dominance
of five LIS countries over the US distribution**

(asymptotic standard errors)

s	CANADA	ITALY	FINLAND	NORWAY	NETHERLANDS $[z^-_s , z^+_s]$
s = 1	27840 (1575)	5340 (161)	12387 (164)	13190 (197)	[2958, 8470] (193) (203)
s = 2	-	6976 (207)	18038 (287)	19708 (389)	[4504, 11095] (486) (389)
s = 3	-	8930 (346)	25092 (562)	28051 (791)	[6128, 13835] (741) (716)
s = 4	-	11042 (471)	33081 (897)	37533 (1299)	[7839, 16530] (1071) (1145)

Table 5

Poverty ranking [based on $D^s(z-x)$] with poverty line equal to half of median income for the USA, Canada, and the Netherlands

(asymptotic standard errors)

x	s = 1			s = 2		
	Most Poverty	Medium Poverty	Least Poverty	Most Poverty	Medium Poverty	Least Poverty
7000	USA 0.012 (0.001)	CAN 0.0070 (0.0006)	NL 0 (-)	USA 0.2 (1.1)	CAN 8.2 (0.8)	NL 0 (-)
6000	USA 0.020 (0.001)	NL 0.012 (0.002)	CAN 0.0108 (0.0007)	USA 26.1 (2.1)	CAN 17.3 (1.4)	NL 3.1 (0.8)
5000	USA 0.030 (0.002)	NL 0.021 (0.002)	CAN 0.016 (0.001)	USA 50.0 (3.4)	CAN 30.4 (2.1)	NL 18.8 (2.5)
4000	USA 0.502 (0.002)	CAN 0.027 (0.001)	NL 0.024 (0.002)	USA 88.9 (5.3)	CAN 51.2 (3.0)	NL 41.0 (4.6)
3000	USA 0.077 (0.003)	CAN 0.041 (0.002)	NL 0.028 (0.003)	USA 152.1 (7.7)	CAN 84.2 (4.2)	NL 67.4 (6.8)
2000	USA 0.160 (0.004)	CAN 0.065 (0.002)	NL 0.035 (0.003)	USA 245 (11)	CAN 136.1 (5.8)	NL 99.1 (9.3)
1000	USA 0.144 (0.004)	CAN 0.088 (0.002)	NL 0.045 (0.004)	USA 372 (14)	CAN 211.6 (7.7)	NL 139 (12)
0	USA 0.181 (0.005)	CAN 0.116 (0.003)	NL 0.067 (0.005)	USA 534 (18)	CAN 312.8 (9.9)	NL 194 (15)

Table 6
CPG Curve [G(p;z)] for a poverty line, z,
equal to half of median income
(asymptotic standard errors)

p	USA	CAN	NL
0.01	80	77.5	62.7
	(11)	(4.7)	(3.4)
0.02	146	129.4	119.6
	(11)	(5.2)	(4.2)
0.03	201	170.8	157.2
	(11)	(5.7)	(8.2)
0.04	247	204.3	178.3
	(11)	(6.2)	(11)
0.05	289	232.6	189
	(12)	(6.6)	(13)
0.06	327	256	194
	(12)	(7.0)	(15)
0.07	362	275.6	194
	(12)	(7.4)	(15)
0.08	392	290.5	194
	(13)	(8.0)	(15)
0.09	419	301.5	194
	(13)	(8.5)	(15)
0.10	444	308.6	194
	(14)	(9.1)	(15)
0.11	465	312.2	194
	(14)	(9.6)	(15)
0.12	483	312.8	194
	(15)	(9.9)	(15)
0.13	499	312.8	194
	(15)	(9.9)	(15)
0.14	511	312.8	194
	(16)	(9.9)	(15)
0.15	521	312.8	194
	(16)	(9.9)	(15)
0.16	528	312.8	194
	(17)	(9.9)	(15)
0.17	532	312.8	194
	(17)	(9.9)	(15)
0.18	534	312.8	194
	(18)	(9.9)	(15)
0.19	534	312.8	194
	(18)	(9.9)	(15)

Table 7

**Relative equality ranking
for dominance of the second order**

Based on estimates of $D^2(\mu \cdot x)/\mu$

(asymptotic standard errors)

x	MOST EQUALITY	SECOND MOST EQUALITY	SECOND LEAST EQUALITY	LEAST EQUALITY
0.25	IT 0.0020 (0.0002)	CAN 0.0026 (0.0002)	NL 0.0053 (0.0005)	USA 0.0063 (0.0003)
0.5	IT 0.0186 (0.0009)	NL 0.0178 (0.0012)	CAN 0.0221 (0.0005)	USA 0.0412 (0.0009)
0.75	NL 0.0707 (0.0026)	CAN 0.0838 (0.0009)	IT 0.0856 (0.0016)	USA 0.120 (0.001)
1,0	NL 0.189 (0.003)	CAN 0.202 (0.001)	IT 0.206 (0.002)	USA 0.245 (0.001)
1.25	NL 0.361 (0.004)	CAN 0.368 (0.001)	IT 0.373 (0.002)	USA 0.408 (0.002)
1.5	NL 0.568 (0.004)	CAN 0.570 (0.002)	IT 0.574 (0.003)	USA 0.601 (0.003)
1.75	CAN 0.792 (0.002)	NL 0.796 (0.005)	IT 0.797 (0.004)	USA 0.815 (0.004)
2.0	CAN 1.027 (0.003)	IT 1.031 (0.006)	NL 1.034 (0.007)	USA 1.042 (0.005)
2.25	CAN 1.268 (0.004)	IT 1.271 (0.007)	NL AND USA 1.277 (0.009)	1.277 (0.006)
2.5	CAN 1.513 (0.005)	IT 1.516 (0.009)	USA 1.518 (0.007)	NL 1.523 (0.012)

Table 8

**Relative equality ranking
for dominance of the third order**

Based on estimates of $D^3(\mu \cdot x)/\mu^2$

(asymptotic standard errors)

x	MOST EQUALITY	SECOND MOST EQUALITY	SECOND LEAST EQUALITY	LEAST EQUALITY
0.25	**IT** 0.00016 (0.00003)	**CAN** 0.00021 (0.000015)	**NL** 0.00046 (0.00006)	**USA** 0.00051 (0.00003)
0.5	**IT** 0.0021 (0.0001)	**CAN** 0.0027 (0.00009)	**NL** 0.0032 (0.0003)	**USA** 0.0056 (0.0002)
0.75	**NL** 0.0129 (0.0006)	**IT** 0.0139 (0.0004)	**CAN** 0.0148 (0.0002)	**USA** 0.0248 (0.0004)
1,0	**NL** 0.044 (0.001)	**IT AND CAN** 0.0493 (0.0007)	0.0493 (0.0004)	**USA** 0.0696 (0.0006)
1.25	**NL** 0.112 (0.002)	**CAN** 0.1197 (0.0008)	**IT** 0.121 (0.001)	**USA** 0.150 (0.001)
1.5	**NL** 0.227 (0.003)	**CAN** 0.236 (0.001)	**IT** 0.239 (0.002)	**USA** 0.276 (0.002)
1.75	**NL** 0.397 (0.005)	**CAN** 0.406 (0.002)	**IT** 0.410 (0.004)	**USA** 0.452 (0.003)
2.0	**NL** 0.626 (0.008)	**CAN** 0.633 (0.004)	**IT** 0.638 (0.006)	**USA** 0.684 (0.006)
2.25	**NL** 0.915 (0.012)	**CAN** 0.920 (0.006)	**IT** 0.926 (0.009)	**USA** 0.974 (0.008)
2.5	**NL** 1.265 (0.0018)	**CAN** 1.268 (0.008)	**IT** 1.274 (0.014)	**USA** 1.323 (0.012)